EIGENVALUE BOUNDS AND MINIMAL SURFACES IN THE BALL

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ABSTRACT. We show that metrics that maximize $\sigma_1 L$ where σ_1 is the first nonzero Steklov eigenvalue and L the boundary length arise as the induced metrics on free boundary minimal surfaces in the unit ball B^n for some n. In the case of the annulus we prove that the critical catenoid, which is the unique free boundary surface of revolution in B^3, is the unique free boundary solution in B^n such that the coordinate functions are first eigenfunctions. We also show that for the Möbius band, there is a unique free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions, given by an explicit S^1 invariant embedding in B^4, the critical Möbius band. For maximizing metrics on oriented surfaces of genus 0 with arbitrarily many boundary components we characterize the limit as the number of boundary components tends to infinity. We also prove multiplicity bounds on σ_1 in terms of the topology, and we give a lower bound on the Morse index for the area functional for free boundary surfaces in the ball.

1. Introduction

If we fix a smooth compact surface M with boundary, we can consider all Riemannian metrics on M with fixed boundary length, say $L(\partial M) = 1$. We can then hope to find a canonical metric by maximizing a first eigenvalue. The eigenvalue problem which turns out to lead to geometrically interesting maximizing metrics is the Steklov eigenvalue; that is, the first eigenvalue σ_1 of the Dirichlet-to-Neumann map on ∂M. In our earlier paper [FS] we made a connection of this problem with minimal surfaces Σ in a euclidean ball which are proper in the ball and which meet the boundary of the ball orthogonally. We refer to such minimal surfaces as *free boundary* surfaces since they arise variationally as critical points of the area among surfaces in the ball whose boundaries lie on ∂B but are free to vary on ∂B. The orthogonality at ∂B makes the area critical for variations that are tangent to ∂B but do not necessarily fix $\partial\Sigma$. Given an oriented surface M of genus γ with $k \geq 1$ boundary components, we let $\sigma^*(\gamma, k)$ denote the supremum of $\sigma_1 L$ taken over all smooth metrics on M. Given a non-orientable surface M we let k denote the number of boundary components and γ denote the genus of its oriented double covering. Thus the Möbius band has $\gamma = 0$ and $k = 1$ while the Klein bottle with a disk removed has $\gamma = 1$ and $k = 1$. We then let $\sigma^\#(\gamma, k)$ denote the supremum of $\sigma_1 L$ taken over all smooth metrics on M.

In this paper we develop the theory in several new directions. First we develop the existence theory of maximizing metrics. An important step in the existence theory is proving bounds on the conformal structure and boundary lengths for metrics that are nearly maximizing. In Section 4 we do this for surfaces of genus 0 with arbitrarily many boundary components. This proof involves showing that the supremum value for $\sigma_1 L$ strictly increases

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when a boundary component is added. It is then shown by delicate constructions of comparison functions that if the conformal structure degenerates for a sequence, then σ_1 for the sequence must be asymptotically bounded above by the supremum for surfaces with fewer boundary components. We also show that the conformal structure is controlled for nearly maximizing metrics on the Möbius band. In a forthcoming paper, we use these results to prove existence of a smooth maximizing metric.

In Section 5 we establish the connection between extremal metrics and minimal surfaces that are free boundary solutions in the ball.

Theorem 1.1. *Let M be a surface. If there exists on M a smooth metric g which maximizes $\sigma_1 L$ over all metrics on M then there is a branched conformal minimal immersion $\varphi : (M, g) \to B^n$ for some $n \geq 3$ by first eigenfunctions so that φ is a σ-homothety from g to the induced metric $\varphi^*(\delta)$ where δ is the euclidean metric on B^n.*

The terminology σ-homothetic means that g is a constant times the induced metric on the boundary.

For the annulus and the Möbius band we are able to uniquely characterize the minimal immersions into B^n given by first eigenfunctions. In the case of the annulus we show that the 'critical catenoid', the unique portion of a suitably scaled catenoid which defines a free boundary surface in B^3, is the unique free boundary minimal annulus such that the coordinate functions are first eigenfunctions. This uniqueness theorem is proved in Section 6.

Theorem 1.2. *If Σ is a free boundary minimal surface in B^n which is homeomorphic to the annulus and such that the coordinate functions are first eigenfunctions, then $n = 3$ and Σ is congruent to the critical catenoid.*

We note that there are multiplicity bounds proven in Section 2 which imply that a maximizing metric in the genus 0 case necessarily lies in B^3. The main idea of the proof is to show that such a surface is S^1 invariant, and this case was analyzed in detail in [FS]. The proof of the uniqueness of the critical catenoid uses several ingredients including an analysis of the second variation of energy and area for free boundary surfaces. We show in Section 2 that certain canonical vector fields reduce the area up to second order; in fact, we show that the Morse index (for area) of any free boundary surface in B^n, which does not split as a product with a line, is at least n. We note that this same class of variations was used by J. D. Moore and T. Schulte [MS] to show that any free boundary submanifold in a convex body is unstable. Our proof seems to use the fact that we are in the ball in important ways. In the uniqueness proof it is necessary to show that the energy has high enough index, so in order to do this we solve a Cauchy-Riemann equation on Σ to add a tangential component to certain normal variations to make them conformal. We note that a systematic study of the relationship between second variation of energy and area was done by N. Ejiri and M. Micallef [EM].

We remark that assuming the existence of a smooth maximizing metric, which we plan to prove in a forthcoming paper, the uniqueness result above would imply the following sharp eigenvalue bound: for any metric on the annulus M,

$$\sigma_1 L \leq (\sigma_1 L)_{cc}$$

with equality if and only if M is σ-homothetic to the critical catenoid; in particular,

$$\sigma^*(0,2) = (\sigma_1 L)_{cc} \approx 4\pi/1.2.$$

In Section 7 we prove the analogous theorem for the Möbius band. We explicitly construct a minimal embedding by first eigenfunctions of a Möbius band into B^4 which defines a free boundary surface. We refer to this surface as the critical Möbius band. We show that this is the only such surface which has an S^1 symmetry group. The following theorem characterizes this metric.

Theorem 1.3. *Assume that Σ is a free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions. Then $n = 4$ and Σ is the critical Möbius band.*

We prove this by extending the argument for the annulus case to show that such a Möbius band is invariant under an S^1 group of rotations of B^n. We then show that $n = 4$ and the surface is the critical Möbius band.

Assuming the existence of a smooth maximizing metric for the Möbius band, which we plan to prove in a forthcoming paper, this would imply that for any metric on the Möbius band M,

$$\sigma_1 L \leq (\sigma_1 L)_{cmb} = 2\pi\sqrt{3}$$

with equality if and only if M is σ-homothetic to the critical Möbius band; in particular,

$$\sigma^{\#}(0,1) = (\sigma_1 L)_{cmb} = 2\pi\sqrt{3}.$$

For a surface of genus 0 and $k \geq 3$ we show in Section 8 that any maximizing metric arises from a free boundary surface in B^3 which is embedded and star-shaped with respect to the origin. We then analyze the limit as k goes to infinity.

Theorem 1.4. *The sequence $\sigma^*(0,k)$ is strictly increasing in k and converges to 4π as k tends to infinity. For each k, if a maximizing metric exists, then it is achieved by a free boundary minimal surface Σ_k in B^3 of area less than 2π. The limit of these minimal surfaces as k tends to infinity is a double disk.*

The proof of this theorem uses properties of the nodal sets of first eigenfunctions (in this case, intersection curves of planes through the origin with the surface) together with a variety of minimal surface methods. The maximizing property of the metrics is used in an essential way to identify the limit as a double plane.

For compact, closed surfaces M there is a question which in certain respects has parallels to the problem we study. This is the question of maximizing $\lambda_1(g)A(g)$ over all smooth metrics on M where λ_1 is the first nonzero eigenvalue. This problem has been resolved for surfaces with non-negative Euler characteristic with contributions by several authors. First, J. Hersch [H] proved that the constant curvature metrics on $\mathbb{S}^2$ uniquely maximize. The connection of this problem with minimal surfaces in spheres was made by P. Li and S. T. Yau [LY] who succeeded in showing that the constant curvature metric on $\mathbb{RP}^2$ uniquely maximizes $\lambda_1 A$ over all smooth metrics on $\mathbb{RP}^2$. It was shown by N. Nadirashvili [N] that the maximizing metric on the torus is uniquely achieved by the flat metric on the 60^0 rhombic torus. Finally, the case of the Klein bottle was handled in a series of papers by Nadirashvili [N] who proved the existence of a maximizing metric, by D. Jakobson, Nadirashvili, and I. Polterovich [JNP] who constructed the maximizing metric (which is $\mathbb{S}^1$ invariant but not

flat), and by A. El Soufi, H. Giacomini, and M. Jazar [EGJ] who proved it is the unique maximizer. The cases of the torus and the Klein bottle are much more difficult technically than the two sphere and the projective plane because they require a difficult theorem of [N] which asserts the existence of a maximizing metric which is the induced metric on a branched conformal minimal immersion into $\mathbb{S}^n$ by first eigenfunctions. As in our existence theorem there are two steps, the first being to control the conformal structure for metrics which are near the maximum, and the second being to prove existence and regularity of the conformal metric. The first part of this theorem has been simplified by A. Girouard [G]. Recently G. Kokarev [K] has undertaken a study of existence and regularity of maximizing measures for $\lambda_1 A$ in a conformal class. He has obtained existence and partial regularity of such measures.

We point out that the case of closed surfaces has to do with minimal immersions into spheres while our theory has to do with minimal immersions into balls with the free boundary condition. A basic result which is important for the theory of closed surfaces is that the conformal transformations of the sphere reduce the area of minimal surfaces. This was shown by Li and Yau [LY] for two dimensional surfaces and extended by A. El Soufi and S. Ilias [EI1] to higher dimensions. The conformal transformations of the ball do not seem to have this property for free boundary minimal surfaces, and this makes our theory, particularly the uniqueness theorems, more difficult. We note that in [FS] it was shown that the conformal transformations of the ball do reduce the boundary length for two dimensional free boundary surfaces, but we do not know if they also reduce the area.

2. Notation and preliminaries

Let (M, g) be a compact k-dimensional Riemannian manifold with boundary $\partial M \neq \emptyset$ and Laplacian Δ_g. Given a function $u \in C^\infty(\partial M)$, let $\hat{u}$ be the harmonic extension of u:

$$\begin{cases} \Delta_g \hat{u} = 0 & \text{on } \ M, \\ \hat{u} = u & \text{on } \partial M. \end{cases}$$

Let η be the outward unit conormal along ∂M. The Dirichlet-to-Neumann map is the map

$$L : C^\infty(\partial M) \to C^\infty(\partial M)$$

given by

$$Lu = \frac{\partial \hat{u}}{\partial \eta}.$$

L is a nonnegative self-adjoint operator with discrete spectrum $\sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots$ tending to infinity. We will refer to these as the Steklov eigenvalues.

Since the constant functions are in the kernel of L, the lowest eigenvalue σ_0 of L is zero. The first nonzero eigenvalue σ_1 of L can be characterized variationally as follows:

$$\sigma_1 = \inf_{u \in C^1(\partial M),\, \int_{\partial M} u = 0} \frac{\int_M |\nabla \hat{u}|^2 \, da}{\int_{\partial M} u^2 \, ds}.$$

The results of this paper concern two dimensional surfaces, and in this case the Steklov problem has a certain conformal invariance which we now elucidate with some terminology. If we have two surfaces (M_1, g_1) and (M_2, g_2), we say that M_1 and M_2 are *σ-isometric* (resp. *σ-homothetic*) if there is a conformal diffeomorphism $\varphi : M_1 \to M_2$ such that the pullback metric $\varphi^*(g_2) = \lambda^2 g_1$ with $\lambda = 1$ (resp. $\lambda = c$ for a constant c) on ∂M_1. It is clear that

if two surfaces are σ-isometric then their Steklov eigenvalues coincide, while if surfaces are σ-homothetic then the normalized eigenvalues $L(\partial M_j)\sigma_i(M_j)$ coincide for $j = 1, 2$ and for all i. We can only hope to characterize surfaces up to σ-homothety by conditions on the Steklov spectrum.

We will need the following coarse upper bound which combines Theorem 2.3 of [FS] with a bound of G. Kokarev [K].

Theorem 2.1. *Let M be a compact oriented surface of genus γ with k boundary components. Let σ_1 be the first non-zero eigenvalue of the Dirichlet-to-Neumann operator on M with metric g. Then*

$$\sigma_1 L(\partial M) \leq \min\{2(\gamma + k)\pi, 8\pi[(\gamma + 3)/2]\}$$

where $[x]$ denotes the greatest integer less than or equal to x.

We will also need the following multiplicity bound for the first eigenvalue on surfaces.

Theorem 2.2. *For a compact connected oriented surface M of genus γ with k boundary components, the multiplicity of σ_i is at most $4\gamma + 2i + 1$.*

Proof. We first show that there for any eigenfunction u on M, the set of points $S = \{p \in \bar{M} : u(p) = 0,\ \nabla u(p) = 0\}$ is finite. It is clear that this set is discrete in the interior of M, and standard unique continuation results imply that if there is point $p \in S \cap \partial M$, then u vanishes to a finite order at p, the point p is isolated in S, and the zero set of u near p consists of a finite number of arcs from p which meet the boundary transversely. To see this we choose conformal coordinates (x, y) centered at p so that M is the upper half plane near $(0, 0)$. The function u is then harmonic and satisfies the boundary condition $u_y = -\sigma u \lambda$ for $y = 0$ where σ is the eigenvalue and λ is a smooth positive function of x. The function $v = e^{\sigma\lambda y} u$ then satisfies the equation $\Delta(e^{-\sigma\lambda y} v) = 0$ together with the homogeneous Neumann boundary condition. This may be written in the form

$$\Delta v + b \cdot \nabla v + cv = 0,\ \ v_y(x, 0) = 0$$

where the coefficient functions are smooth and bounded up to the boundary. We then extend v by even reflection to a full neighborhood of $(0, 0)$ and observe that the extended function $\hat{v}$ satisfies

$$\Delta\hat{v} + \hat{b} \cdot \nabla\hat{v} + \hat{c}\hat{v} = 0$$

where $\hat{b}$ is the odd extension of b and $\hat{c}$ the even extension of c. The coefficients $\hat{b}$ and $\hat{c}$ are bounded and therefore we may apply unique continuation results (see [HW]) to assert that v and hence u vanishes to finite order at $(0, 0)$. From the boundary condition on u it follows that the leading order homogenous harmonic polynomial $P(x, y)$ in the Taylor expansion of u at $(0, 0)$ is even under reflection across the x-axis and satisfies $P_y(x, 0) = 0$. It follows that $P_x(x, 0) \neq 0$ for $x \neq 0$ and the zero set of P consists of lines through the origin which are symmetric under reflection across the x-axis and which do not contain the x-axis. The conclusion for the zero set of u then follows.

Let φ be an i-th eigenfunction. Suppose $p \in M$ and $\varphi(p) = 0$. First we show that the order of vanishing of φ at p is less than or equal to $2\gamma+i$. To see this, suppose the order of vanishing of φ is d. By [Ch] (Theorem 2.5), the nodal set of φ consists of a number of C^2-immersed one dimensional closed submanifolds, which meet at a finite number of points. Therefore, the nodal set of φ consists of a finite number of immersed circles and arcs between boundary

components. Since the sign of an eigenfunction changes around any of its zeros, there are an even number of arcs meeting any given boundary curve of M. By gluing a disk on each boundary component of M and deforming the disk to a point, we can view M as a compact surface S of genus γ with k points $p_1, \dots, p_k$ removed, where each point is identified with a boundary curve of M. Since the nodal set of a spherical harmonic of order d in $\mathbb{R}^2$ consists of d lines passing through the origin, there exist injective piecewise C^1 maps $\Phi_j : S^1 \to S$, $j = 1, \dots, d$ such that $\Phi_j(S^1) \cap \Phi_l(S^1)$, $j \neq l$, consists of a finite number of points, and $\Phi_j(S^1) \subset \varphi^{-1}(0) \cup \{p_1, \dots, p_k\}$, $j = 1, \dots, d$. If $d > 2\gamma + i$, then there exist $n_1, \dots, n_{2\gamma+i+1} \in \mathbb{Z}$ not all zero such that the homology class of S represented by $\sum_{j=1}^{2\gamma+i+1} n_j \Phi_j$ is zero. It follows from the argument of [Ch] (Lemma 3.1) that $S \setminus \Phi_1(S^1) \cup \ldots \cup \Phi_d(S^1)$ has at least $i+2$ connected components. Therefore, $M \setminus \varphi^{-1}(0)$ has at least $i+2$ components. But by the nodal domain theorem for Steklov eigenfunctions (see [KS]) there can be at most $i+1$ connected components of $M \setminus \varphi^{-1}(0)$. Therefore the order of vanishing of φ is less than or equal to $2\gamma + i$.

Now, arguing as in [Be] and [Ch], we show that the multiplicity m_i of σ_i is at most $4\gamma + 2i + 1$. Let $\varphi_1, \dots, \varphi_{m_i}$ be a basis for the i-th eigenspace. Let (x, y) be local conformal coordinates near $p \in M$. Consider the system of equations

$$\sum_{j=1}^{m_i} a_j \frac{\partial^s \varphi_j}{\partial x^t \partial y^{s-t}}(p) = 0, \quad s = 0, \dots, 2\gamma + i + 1, \;\; 0 \le t \le s \tag{2.1}$$

This is a system of $(2\gamma + i + 1)(2\gamma + i + 2)/2$ equations in m_i unknowns, $a_1, \dots, a_{m_i}$. Since $\Delta \varphi_j = 0$, we have for any integer q, and $0 \le t \le q$:

$$\frac{\partial^{q+2} \varphi_j}{\partial x^{l+2} \partial y^{q-l}}(p) + \frac{\partial^{q+2} \varphi_j}{\partial x^{l} \partial y^{q-l+2}}(p) = 0.$$

Using this, it can then be shown that $(2\gamma + i - 1)(2\gamma + i)/2$ of the equations (2.1) follow from the other ones. Therefore, to solve (2.1), it suffices to solve the remaining $4\gamma + 2i + 1$ equations in m_i unknowns. If $m_i > 4\gamma + 2i + 1$, then this homogeneous system of linear equations has a nontrivial solution. But then $\varphi = \sum_{j=1}^{m_i} a_j \varphi_j$ would be an i-th eigenfunction that vanishes to order $2\gamma + i + 1$, a contradiction. Therefore, $m_i \le 4\gamma + 2i + 1$. □

For non-orientable surfaces, we have:

Theorem 2.3. *For a compact connected non-orientable surface M with k boundary components and Euler characteristic $\chi(M)$, the multiplicity of σ_i is at most $4(1-\chi(M)-k)+4i+3$.*

Proof. We can view M as a domain in a compact surface S of Euler characteristic $\chi(M)+k$. Then, by an argument similar to [Be], Theorem 2.2, the multiplicity of σ_i is at most $4\tilde{\gamma}+4i+3$, where $\tilde{\gamma} = 1 - \chi(M) - k$ is the genus of the orientable double cover of S. □

3. The Morse index of free boundary solutions in the ball

Following [FS] we will say that a minimal submanifold Σ, properly immersed in a domain $\Omega \subset \mathbb{R}^n$, is a *free boundary solution* if the outward unit normal vector of $\partial\Sigma$ (the conormal vector) agrees with the outward unit normal to $\partial\Omega$ at each point of $\partial\Sigma$. If φ is an isometric minimal immersion of M into the unit ball B^n such that $\Sigma = \varphi(M)$ is a free boundary solution, then the coordinate functions $\varphi^1, \dots, \varphi^n$ are Steklov eigenfunctions with eigenvalue

1. In this section we show that certain normal deformations decrease volume to second order. For a free boundary solution there is a Morse index for deformations which preserve the domain Ω but not necessarily the boundary of Σ. In general this is larger than the Morse index for deformations which fix the boundary. For example, the Morse index of the critical catenoid is at least three for deformations which fix the ball, while it is zero for deformations which fix the boundary of Σ.

We consider a free boundary submanifold Σ^k in the ball B^n. For a normal variation W we have the index form for area given by

$$S(W,W) = \int_\Sigma (|DW|^2 - |A^W|^2)\ da - \int_{\partial\Sigma} |W|^2\ ds$$

where A denotes the second fundamental form of Σ.

Theorem 3.1. *If Σ^k is a free boundary solution in B^n and $v \in \mathbb{R}^n$, then we have*

$$S(v^\perp, v^\perp) = -k \int_\Sigma |v^\perp|^2\ da.$$

If Σ is not contained in a product $\Sigma_0 \times \mathbb{R}$ where Σ_0 is a free boundary solution, then the Morse index of Σ is at least n. In particular, if $k = 2$ and Σ is not a plane disk, its index is at least n.

Proof. Fix a point $x_0 \in \partial\Sigma$ and choose local orthonormal frames $e_1, \ldots, e_k$ tangent to Σ^k, where $e_k = x$ along $\partial\Sigma$, and $\nu_1, \ldots, \nu_{n-k}$ normal to Σ^k such that $(D\nu_\alpha)^\perp = 0$ at x_0. If $h^\alpha_{ij} = (D_{e_i} e_j)\cdot\nu_\alpha$ is the second fundamental form in this basis, then we have $h^\alpha_{ik} = (D_{e_i} x)\cdot\nu_\alpha = e_i \cdot \nu_\alpha = 0$ for $i < k$. Therefore,

$$D_x \nu_\alpha = \sum_{i=1}^k (D_x \nu_\alpha \cdot e_i) e_i = -h^\alpha_{kk} x.$$

Without loss of generality we may assume $|v| = 1$, and we compute $S(v^\perp, v^\perp)$. Observing that $v^\perp$ is a Jacobi field we have

$$S(v^\perp, v^\perp) = \int_{\partial\Sigma} [v^\perp \cdot D_x v^\perp - |v^\perp|^2]\ ds.$$

We calculate the first term using the observation above and the minimality of Σ

$$\begin{aligned} v^\perp \cdot D_x v^\perp &= v^\perp \cdot D_x \left[\sum_{\alpha=1}^{n-k} (v\cdot\nu_\alpha)\nu_\alpha\right] = -v^\perp \cdot \left[\sum_{\alpha=1}^{n-k} (h^\alpha_{kk}(v\cdot x)\nu_\alpha + h^\alpha_{kk}(v\cdot\nu_\alpha)x)\right] \\ &= \sum_{\alpha=1}^{n-k}\sum_{i=1}^{k-1} h^\alpha_{ii}(v\cdot x)(\nu_\alpha \cdot v^\perp). \end{aligned}$$

Now if we let $v_0 = v - (v\cdot x)x$ be the component of v tangent to S^{n-1}, then we have $div_{\partial\Sigma}(v_0) = -(k-1)v\cdot x$. On the other hand $v_0 = v_1 + \sum_{\alpha=1}^{n-k}(v\cdot\nu_\alpha)\nu_\alpha$ where v_1 is the component of v tangent to $\partial\Sigma$, and so $div_{\partial\Sigma}(v_0) = div_{\partial\Sigma}(v_1) - \sum_{i=1}^{k-1}\sum_{\alpha=1}^{n-k}(v\cdot\nu_\alpha)h^\alpha_{ii}$. Putting this information into the index form we find

$$S(v^\perp, v^\perp) = \int_{\partial\Sigma} [(v\cdot x)(div_{\partial\Sigma}(v_1) + (k-1)(v\cdot x)) - |v^\perp|^2]\ ds.$$

We apply the divergence theorem to the first term to get

$$S(v^\perp, v^\perp) = \int_{\partial\Sigma} [-|v_1|^2 + (k-1)(v\cdot x)^2 - |v^\perp|^2]\ ds.$$

Since $|v_1|^2 = 1 - (v\cdot x)^2 - |v^\perp|^2$ we have

$$S(v^\perp, v^\perp) = \int_{\partial\Sigma} [-1 + k(v\cdot x)^2]\ ds.$$

Now we consider the vector field $V = x - k(v\cdot x)v$ and apply the first variation formula on Σ

$$\int_\Sigma div_\Sigma V\ da = \int_{\partial\Sigma} V\cdot x\ ds.$$

Direct computation gives $div_\Sigma V = k(1-|v^t|^2) = k|v^\perp|^2$ where v^t denotes the tangential part of v. Putting this into the formula above we get

$$S(v^\perp, v^\perp) = -\int_{\partial\Sigma} V\cdot x\ ds = -k\int_\Sigma |v^\perp|^2\ da$$

as desired.

If there is a $v \neq 0$ such that $v^\perp \equiv 0$ on Σ, then v lies in the tangent plane to Σ at each point, and Σ is contained in the product $\Sigma_0^{k-1} \times \mathbb{R}$. Σ_0 is the intersection of Σ with the hyperplane through the origin orthogonal to v, and hence is a free boundary solution. Therefore, if Σ is not contained in a product $\Sigma_0 \times \mathbb{R}$ where Σ_0 is a free boundary solution, then the Morse index of Σ is at least n. In particular, if $k = 2$ and Σ is not a plane disk, its index is at least n. □

4. Properties of metrics with lower bounds on $\sigma_1 L$

In this section we take up the existence question for metrics which maximize $\sigma_1 L$. For an oriented surface M with genus γ and $k \geq 1$ boundary components we define the number $\sigma^*(\gamma, k)$ to be the supremum of $\sigma_1(g)L_g(\partial M)$ taken over all smooth metrics on M. Theorem 2.1 tells us that

$$\sigma^*(\gamma, k) \leq \min\{2(\gamma+k)\pi, 8\pi[(\gamma+3)/2]\}.$$

Weinstock's theorem implies that $\sigma^*(0,1) = 2\pi$, and our analysis of rotationally symmetric metrics on the annulus [FS] implies that $\sigma^*(0,2)$ is at least the value for the critical catenoid, so $\sigma^*(0,2) > 2\pi$.

There are two general ways in which metrics can degenerate in our problem. The first is that the conformal structure might degenerate and the second is that the boundary arclength measure might become singular even though the conformal class is controlled. We will need the following general result which limits the way in which the boundary measures can degenerate provided the conformal structures converge and σ_1 is not too small. If we choose a metric g on M and a measure μ with a smooth density relative to the arclength measure of g on ∂M, then $\sigma_1(g,\mu)$ is defined as the first Steklov eigenvalue of any metric conformal to g with arclength measure μ on ∂M. To be clear we have

$$\sigma_1(g,\mu) = \inf\{\frac{\int_M |\nabla\hat{u}|^2\ da_g}{\int_{\partial M} u^2\ d\mu} : \int_{\partial M} u\ d\mu = 0\}.$$

By the weak* compactness of measures, if we take a sequence of probability measures μ_i on ∂M, then there is a subsequence which converges to a limit probability measure μ.

Proposition 4.1. *Assume that we have a sequence of metrics g_i converging in C^2 norm to a metric g on M, and a sequence of smooth probability measures μ_i converging in the weak* topology to a measure μ. Assume that there is a $\lambda > 2\pi$ so that $\sigma_1(g_i, \mu_i) \geq \lambda$ for each i. It then follows that μ has no point masses.*

Proof. Assume on the contrary there is a point $p \in \partial M$ with $a = \mu(\{p\}) > 0$. We consider two cases: First assume that $a < 1$. In this case we will show that $\sigma_1(g_i, \mu_i)$ converges to 0. We can find a sufficiently small open interval I in ∂M containing p so that $b = \mu(\partial M \setminus I) > 0$. We then choose a smooth function $u \leq 1$ supported in I with $u(p) = 1$ and so that the Dirichlet integral of the harmonic extension $\hat{u}$ is arbitrarily small (relative to the the limit metric g). We then consider the function $\varphi = u - \bar{u}$ where $\bar{u}$ is the average of u with respect to μ. The harmonic extension $\hat{\varphi}$ has the same Dirichlet integral as that of u, and we have $\bar{u} \leq \mu(I) = 1 - b$, and thus $\varphi(p) \geq b$, and so we have

$$\int_{\partial M} \varphi^2 \, d\mu \geq ab^2 > 0.$$

It follows that the Raleigh quotient of φ with respect to (g_i, μ_i) becomes arbitrarily small for large i contradicting the lower bound on $\sigma_1(g_i, \mu_i)$.

The second and more difficult case is when $a = 1$, and the support of μ is p. In this case we show that $\limsup_{i \to \infty} \sigma_1(g_i, \mu_i) \leq 2\pi$ in contradiction to our assumption. To see this we consider a disk U in M whose boundary intersects ∂M in an interval I about p. We define a measure ν_i on ∂U to be equal to μ_i on I and to be zero on $\partial U \setminus I$. We then take a conformal map F_i from U to the unit disk D for which $\int_{\partial U} F_i \, d\nu_i = 0$. Since the measures ν_i are converging to a point mass at p, it follows that the maps F_i outside a neighborhood of p are arbitrarily close to a point of ∂D for i large. By composing F_i with a rotation we may assume that this point is $(1, 0)$, so we have arranged that the F_i converge to the constant map $(1, 0)$ on compact subsets of $\bar{U} \setminus \{p\}$; in particular, this is true near $\partial U \setminus I$. We may then extend F_i to a Lipschitz map $\hat{F}_i : M \to \mathbb{R}^2$ which maps the complement of a neighborhood of $\bar{U}$ to the point $(1, 0)$ and such that the Dirichlet integral of $\hat{F}_i$ converges to 2π, the Dirichlet integral of F_i. The average of $\hat{F}_i$ on ∂M with respect to μ_i converges to 0, so we may use $\hat{F}_i$ minus its average as comparison functions as in the Weinstock proof to conclude that $\limsup_{i \to \infty} \sigma_1(g_i, \mu_i) \leq 2\pi$. This contradiction completes the proof. □

We now prove a result on the strict monotonicity of σ^* when the number of boundary components is increased.

Proposition 4.2. *For any γ and k we have $\sigma^*(\gamma, k+1) \geq \sigma^*(\gamma, k)$. If $\sigma^*(\gamma, k)$ is achieved by a smooth metric for some γ and k it then follows that $\sigma^*(\gamma, k+1) > \sigma^*(\gamma, k)$.*

Proof. The weak inequality follows because for any $\epsilon > 0$ we can choose a smooth metric g on a surface M of genus γ with k boundary components with $L_g(\partial M) = 1$ and $\sigma_1(g) \geq \sigma^*(\gamma, k) - \epsilon$. If we cut out a small disk from M we get a surface of genus γ with $k+1$ boundary components, and it is shown below that this new surface with the restricted metric g has $\sigma_1 L$ very close to that of (M, g) when the disk is small. It follows that $\sigma^*(\gamma, k+1) \geq \sigma^*(\gamma, k) - 2\epsilon$ for any $\epsilon > 0$ and hence $\sigma^*(\gamma, k+1) \geq \sigma^*(\gamma, k)$.

For the strict inequality, let M be a surface of genus γ with k boundary components and let g be a metric with $L_g(\partial M) = 1$ and $\sigma_1(g) = \sigma^*(\gamma, k)$. Let p be a point of M, and without loss of generality assume that g is flat in a neighborhood of p with euclidean coordinates x, y centered at p. Let D_ϵ be the disk of radius ϵ centered at p, and let M_ϵ be the surface $M \setminus D_\epsilon$ which has genus γ and $k+1$ boundary components. We let g_ϵ denote the metric g restricted to M_ϵ and $\sigma_1(g_\epsilon)$ the corresponding first Steklov eigenvalue. Our goal is to show that

$$\sigma_1(g_\epsilon) \geq \sigma_1(g) - c\epsilon^2 \tag{4.1}$$

for a constant c. It then follows that

$$L_{g_\epsilon}(M_\epsilon)\sigma_1(g_\epsilon) = (1 + 2\pi\epsilon)\sigma_1(g_\epsilon) \geq (1 + 2\pi\epsilon)(\sigma^*(\gamma, k) - c\epsilon^2) > \sigma^*(\gamma, k)$$

for ϵ small and positive. It will then follow that $\sigma^*(\gamma, k+1) > \sigma^*(\gamma, k)$.

To prove (4.1) we first show that $\sigma_1(g_\epsilon)$ converges to $\sigma_1(g)$ as ϵ goes to zero. We let u_ϵ be a first eigenfunction of g_ϵ with $\int_{\partial M_\epsilon} u_\epsilon^2\ ds = 1$. We show that there is a sequence ϵ_i tending to 0 such that the corresponding sequence u_i converges to u in C^2 norm on compact subsets of $M \setminus \{p\}$, u is a first eigenfunction of g with $\int_{\partial M} u^2\ ds = 1$, and the Dirichlet integrals converge. The main point is to show that $\int_{\partial D_\epsilon} u_\epsilon^2\ ds$ tends to zero. This can be seen by taking a harmonic function h on M which is a Green's function with pole at p satisfying $h(x) - \log(|x|)$ is bounded near p and $h = 0$ on ∂M. If we apply Green's formula using the identity $1/2\Delta u_\epsilon^2 = |\nabla u_\epsilon|^2$ on M_ϵ we find

$$\int_{M_\epsilon} h|\nabla u_\epsilon|^2\ da = 1/2 \int_{\partial M_\epsilon} (h\frac{\partial u_\epsilon^2}{\partial \eta} - u_\epsilon^2 \frac{\partial h}{\partial \eta})\ ds.$$

Using the boundary conditions, the bound on $\sigma_1(g_\epsilon)$ (from the coarse upper bound), and the bounds on h we see that

$$\int_{\partial D_\epsilon} u_\epsilon^2 \leq c\epsilon|\log \epsilon|$$

for ϵ small. Since $\sigma_1(g_\epsilon)$ is bounded we may choose a sequence ϵ_i tending to 0 such that $\sigma_1(g_i)$ converges to a number $\hat{\sigma}$, and so that the eigenfunctions u_i converge to u in C^2 norm on compact subsets of $M \setminus \{p\}$. It then follows from above that $\int_{\partial M} u^2\ ds = 1$. Clearly u is harmonic on $M \setminus \{p\}$ and since it has finite Dirichlet integral it is harmonic on M. Integration by parts implies $\hat{\sigma} = E(u)$ while $\sigma_1(g_i) = E_{M_i}(u_i)$, so we see that the Dirichlet integrals converge and it follows that $\hat{\sigma} = \sigma_1(g)$. Since the limit is unique we see that $\sigma_1(g_\epsilon)$ converges to $\sigma_1(g)$.

We now show that $\int_{\partial D_\epsilon} u_\epsilon^2\ ds \leq c\epsilon^3$ for a constant c and ϵ small. We first estimate $\int_{\partial D_\epsilon} u\ ds$. This we do by using a Green's function h for M with pole at p and boundary condition $\frac{\partial h}{\partial \eta} - \sigma_1(g_\epsilon)h = 0$. There may be a difficulty finding such a function with a single pole if $\sigma_1(g_\epsilon)$ is a Steklov eigenvalue, but if we choose enough additional poles and take linear combinations of δ functions to make the sum orthogonal to the eigenfunctions corresponding to eigenvalues which are near $\sigma_1(g)$ we can find h. We can choose the additional poles to lie on the zero set of u_ϵ to make the following argument work. We use Green's formula on M_ϵ to obtain

$$\int_{\partial D_\epsilon} (\frac{\partial h}{\partial r} + \sigma_1(g_\epsilon)h)u_\epsilon\ ds = 0$$

where all other terms vanish because of the choice of the poles of h and the boundary conditions. Since $h(x) = \log(|x|) + \alpha + h_1(x)$ where α is a constant and h_1 a smooth harmonic function with $h_1(0) = 0$, we have the bound

$$|\int_{\partial D_\epsilon} u_\epsilon \, ds| \leq c\epsilon \int_{\partial D_\epsilon} |u_\epsilon| \, ds.$$

We may do a Laurent-type decomposition for u_ϵ

$$u_\epsilon(r, \theta) = a \log r + b + v(r, \theta) + w(r, \theta)$$

where v is smooth and harmonic for $r < r_0$ (r_0 fixed) with $v(0) = 0$, and w is smooth and harmonic for $r \geq \epsilon$ with $|w(x)| \leq c|x|^{-1}$ for $|x|$ large. The boundary condition then implies

$$v_r - \sigma_1(g_\epsilon)v + w_r - \sigma_1(g_\epsilon)w = 0$$

for $r = \epsilon$. This clearly implies that $|w_r - \sigma_1(g_\epsilon)w|$ is bounded on ∂D_ϵ. It is a standard fact for harmonic functions in the plane that the quantity $\int_{\partial D_\rho} |w_T|^2 - |w_r|^2 \, ds$ is independent of ρ where T denotes the unit tangent vector to ∂D_r. In our case this integral must be zero for all $\rho \geq \epsilon$ since the integrand decays like ρ^{-4}. Therefore we have from above

$$\int_{\partial D_\epsilon} |w_T|^2 \, ds = \int_{\partial D_\epsilon} |w_r|^2 \, ds \leq c \int_{\partial D_\epsilon} w^2 \, ds + c\epsilon.$$

Since the integral of w around ∂D_ϵ is 0, we have by the standard Poincaré inequality on a circle

$$\int_{\partial D_\epsilon} w^2 \, ds \leq \epsilon^2 \int_{\partial D_\epsilon} |w_T|^2 \, ds.$$

Combining this with the inequality above and absorbing the squared L^2 norm of w back we obtain $\int_{\partial D_\epsilon} w^2 \, ds \leq c\epsilon^3$. Since this inequality clearly holds for v, we have

$$\int_{\partial D_\epsilon} u_\epsilon^2 \, ds \leq 2\pi\epsilon\big((2\pi\epsilon)^{-1} \int_{\partial D_\epsilon} u_\epsilon \, ds\big)^2 + c\epsilon^3 \leq c\epsilon^2 \int_{\partial D_\epsilon} u_\epsilon^2 \, ds + c\epsilon^3.$$

This implies $\int_{\partial D_\epsilon} u_\epsilon^2 \, ds \leq c\epsilon^3$ as claimed.

To complete the proof of (4.1) we extend u_ϵ to a function $\hat{u}$ on M by defining $\hat{u}(r, \theta) = \frac{r}{\epsilon} u_\epsilon(\epsilon, \theta)$ on D_ϵ. We then have $|\nabla \hat{u}|^2 = \epsilon^{-2} u_\epsilon^2(\epsilon, \theta) + (u_\epsilon)_T^2(\epsilon, \theta)$ where T is the unit tangent to ∂D_ϵ. From the bounds we obtained above we can see that $\int_{D_\epsilon} |\nabla \hat{u}|^2 \, da \leq c\epsilon^2$. We now use $v = \hat{u} - \int_{\partial M} u \, ds$ as a comparison function for the eigenvalue problem on (M, g). This gives

$$\sigma_1(g)(\int_{\partial M} u_\epsilon^2 \, ds - (\int_{\partial M} u_\epsilon \, ds)^2) \leq \int_M |\nabla \hat{u}|^2 \, da \leq \int_{M_\epsilon} |\nabla u_\epsilon|^2 \, da + c\epsilon^2 = \sigma_1(g_\epsilon) + c\epsilon^2.$$

Finally we have $\int_{\partial M} u_\epsilon \, ds = \int_{\partial D_\epsilon} u_\epsilon \, ds$ and from above we see that this quantity is bounded by $c\epsilon^2$. Combining these we see that $\sigma_1(g_\epsilon) \geq \sigma_1(g) - c\epsilon^2$. This completes the proof of Proposition 4.2. □

We will now prove the main theorem of this section which says roughly that for any metric g on a surface with genus 0 and k boundary components with $\sigma_1 L$ strictly above $\sigma^*(0, k-1)$, the conformal structure induced by g lies in a compact subset of the moduli space of conformal structures. In order to measure this notion, we note that, given a smooth surface M of genus γ with k boundary components (except $\gamma = 0$, $k = 1, 2$) and a metric

g on M, there is a unique hyperbolic metric g_0 in the conformal class of g such that the boundary curves are geodesics. This may be obtained by taking the hyperbolic metric on the doubled conformal surface $\tilde{M}$ and restricting it to M. We can measure the conformal class by considering the injectivity radius $i(g_0)$ on $\tilde{M}$; that is, compact subsets of the moduli space are precisely those g_0 with $i(g_0)$ bounded below by a positive constant.

The case $\gamma = 0$ and $k = 1$ is handled by Weinstock's theorem, and for the case $\gamma = 0$ and $k = 2$ the doubled surface is a torus, and thus g_0 must be taken to be a flat metric which we normalize to have area 1. Since the proofs for the annulus and Möbius band are slightly different from the case $k \geq 3$, we handle them separately in the following Proposition. We fix a topological annulus M and we consider metrics g on M. For the Möbius band we consider the oriented double covering. We let $L_1 \geq L_2$ denote the lengths of the boundary curves with $L = L_1 + L_2$, and we let $\alpha = L_1/L_2 \geq 1$ denote the ratio. The Riemannian surface (M, g) is conformally equivalent to $[-T, T] \times S^1$ for a unique $T > 0$. We have the following result.

Proposition 4.3. *Let M be an annulus or a Möbius band. Given $\delta > 0$, there are positive numbers α_0 and β_0 depending on δ such that if g is any metric on M with $\sigma_1(g)L \geq 2\pi(1+\delta)$, then $\alpha \leq \alpha_0$ and $\beta_0^{-1} \leq T \leq \beta_0$.*

Proof. We first handle the case of the annulus. By scaling the metric we may assume that $L = 1$, so we have $L_1 = \alpha/(1+\alpha)$ and $L_2 = (1+\alpha)^{-1}$. We take a conformal diffeomorphism from M into the unit disk D which takes Γ_1 to the unit circle and whose image is a rotationally symmetric annulus. We then compose with a conformal diffeomorphism of the disk to obtain $\varphi : M \to D$ with $\int_{\Gamma_1} \varphi \, ds = 0$. We let $\bar{\varphi}$ denote the mean value of φ over ∂M and observe

$$|\bar{\varphi}| = |\int_{\Gamma_2} \varphi \, ds| < L_2 = \frac{1}{1+\alpha}.$$

Therefore we have $\sigma_1 \int_{\partial M} |\varphi - \bar{\varphi}|^2 \leq \int_M |\nabla \varphi|^2 \, da < 2\pi$. This implies $\sigma_1(1 - |\bar{\varphi}|^2) < 2\pi$ and using the lower bound on σ_1 and the upper bound on $\bar{\varphi}$ we obtain $(1+\delta)(1-(1+\alpha)^{-2}) < 1$ which in turn implies $\alpha \leq \alpha_0$ where $\alpha_0 = \sqrt{(1+\delta)/\delta}$.

To obtain an upper bound on T we consider the linear function $u = (L_1 - L_2) + T^{-1}t$ on $[-T, T] \times S^1$, and observe that $\int_{\partial M} u \, ds = 0$. We thus have

$$\sigma_1 \int_{\partial M} u^2 \, ds \leq \int_M |\nabla u|^2 \, da = 4\pi T^{-1}.$$

Using $\sigma_1 \geq 2\pi$ and $\int_{\Gamma_2} u^2 \, ds \geq L_2$ (since $u \geq 1$ when $t = T$) we have $T \leq 2L_2^{-1} \leq \beta_0$ where we may take $\beta_0 = 2(1 + \alpha_0)$.

The lower bound on T is more subtle. We write the boundary measure as $\lambda(t, \theta) \, d\theta$ where λ is defined at $t = -T$ and $t = T$. Now let us suppose that we normalize $L = 1$ and we define a function $\hat{\lambda}$ on S^1 to be $\hat{\lambda}(\theta) = \lambda(-T, \theta) + \lambda(T, \theta)$. We then have $L = 1 = \int_{S^1} \hat{\lambda} \, d\theta$. There is an interval of length $\pi/2$ which carries at least $1/4$ of the length, so let's choose the point $\theta = 0$ so that $\int_{-\pi/4}^{\pi/4} \hat{\lambda} \, d\theta \geq 1/4$. We now let $L_0 = \int_{\pi/2}^{3\pi/2} \hat{\lambda} \, d\theta$, and we let $u(\theta)$ be a smooth 2π-periodic function which satisfies $u(\theta) = 1$ for $|\theta| \leq \pi/4$, $u(\theta) = -1$ for $\pi/2 \leq \theta \leq 3\pi/2$, and $\int_{S^1} (u')^2 \, d\theta \leq c_1$ for a fixed constant c_1. Using the test function u it is easy to derive the bound $\sigma_1 L_0 \leq c_2 T$ for a fixed constant c_2. We thus obtain a contradiction unless $L_0 \leq c_3 T$ with $c_3 = (2\pi)^{-1} c_2$.

We now consider the case $L_0 \leq c_3 T$ where T is small. In this case we consider the rectangle $R = [-T, T] \times [-3\pi/4, 3\pi/4]$. We take the arclength measure to be 0 on the ends $\theta = -3\pi/4$ and $\theta = 3\pi/4$. We then let $\varphi : R \to D$ be a balanced conformal diffeomorphism in the sense that $\int_{\partial R} \varphi \; d\mu = 0$ where μ is the boundary measure that we have described. We then have $\int_R |\nabla\varphi|^2 \; da = 2\pi$, and by Fubini's theorem there exist $\theta_1 \in [\pi/2, 5\pi/8]$ and $\theta_2 \in [-5\pi/8, -\pi/2]$ such that for $i = 1, 2$ we have the bound

$$\int_{-T}^{T} |\nabla\varphi|^2(t, \theta_i) \; dt \leq c$$

for a fixed constant c. Let $\alpha_i(t) = \varphi(t, \theta_i)$, and we have the bound

$$L(\alpha_i) = \int_{-T}^{T} |\frac{\partial\varphi(t, \theta_i)}{\partial t}| \; dt \leq \int_{-T}^{T} |\nabla\varphi|(t, \theta_i) \; dt \leq (2T)^{1/2} (\int_{-T}^{T} |\nabla\varphi|^2(t, \theta_i) \; dt)^{1/2},$$

and thus $L(\alpha_i) \leq cT^{1/2}$ for a constant c. Now each curve α_i divides the unit disk into a large region and a small region. Because of the balancing condition and the fact that most of the arclength is in the subset of R with $|\theta| \leq \pi/2$, it follows that the subrectangles $\theta_1 \leq \theta \leq 3\pi/4$ and $-3\pi/4 \leq \theta \leq \theta_2$ map into the small region. By the isoperimetric inequality it follows that the areas of these small regions are bounded by constants times T. Therefore we have the bounds

$$\int_{[-T,T]\times[\theta_1, 3\pi/4]} |\nabla\varphi|^2 \; da \leq cT, \quad \int_{[-T,T]\times[-3\pi/4, \theta_2]} |\nabla\varphi|^2 \; da \leq cT.$$

Another application of Fubini's theorem now gives a $\theta_3 \in [5\pi/8, 3\pi/4]$ and $\theta_4 \in [-3\pi/4, -5\pi/8]$ with

$$\int_{-T}^{T} |\nabla\varphi|^2(t, \theta_i) \; dt \leq cT$$

for $i = 3, 4$. We can now define a Lipschitz map $\hat{\varphi} : M \to D$ on $[-T, T] \times [-\pi, \pi] = M$ such that $\hat{\varphi} = \varphi$ for $\theta_4 \leq t \leq \theta_3$, $\hat{\varphi}(t, -\pi) = 0$, and such that $\hat{\varphi}$ on the intervals $[\theta_3, \pi]$ and $[-\pi, \theta_4]$ changes linearly in the θ variable from 0 to φ. By the above bounds, the energy of $\hat{\varphi}$ in these regions is bounded by a constant times T, and therefore

$$\int_M |\nabla\hat{\varphi}|^2 \; da \leq 2\pi + cT.$$

From the fact that $\hat{\varphi}$ is bounded and the bounds on the arclength in the region on which φ was modified, we see

$$|\int_{\partial M} \hat{\varphi} \; ds| \leq cT, |\int_{\partial M} \hat{\varphi}^2 \; ds - 1| \leq cT.$$

We then use the components of $\hat{\varphi} - \int_{\partial M} \hat{\varphi} \; ds$ as comparison functions for $\sigma_1(g)$, and we have

$$\sigma_1(g)\big(\int_{\partial M} |\hat{\varphi}|^2 \; ds - |\int_{\partial M} \hat{\varphi} \; ds|^2\big) \leq \int_M |\nabla\hat{\varphi}|^2 \; da.$$

Therefore we conclude that $\sigma_1(g) \leq 2\pi + cT$ which is a contradiction if T is small.

In case M is a Möbius band, we let $[-T, T] \times S^1$ be the oriented double covering so that M is the quotient gotten by identifying (t, θ) with $(-t, \theta + \pi)$. In this case $L_1 = L_2 = L(\partial M)$ which we normalize to be 1. The arguments follow very much along the lines above for the annulus. To get an upper bound on T we take a conformal map φ from $[T/2, T] \times S^1$ to a

concentric annulus in the plane $D \setminus D_r$ noting that $r = e^{-T/2}$. We then balance φ using a conformal automorphism of D making $\int_{\{T\}\times S^1} \varphi\, ds = 0$. This balanced φ takes the circle $\{3T/4\} \times S^1$ to a circle with radius at most $e^{-T/4}$. In particular it follows that the Dirichlet integral of φ between $T/2$ and $3T/4$ is bounded by a constant times $e^{-T/2}$ (the area enclosed by the circle). Thus by Fubini's theorem we can find $T_1 \in [T/2, 3T/4]$ such that

$$\int_{\{T_1\}\times S^1} |\nabla\varphi|^2\, d\theta \le cT^{-1}$$

where we have made a coarse estimate. We can then extend φ to a map $\hat{\varphi} : [0,T] \times S^1$ by linearly interpolating in the t variable from φ to 0 on the interval $[0, T_1]$ with the properties that $\hat{\varphi}(0,\theta) = 0$, $\int_{\{T\}\times S^1} |\hat{\varphi}|^2\, ds = 1$, $\int_{\{T\}\times S^1} \hat{\varphi}\, ds = 0$, and

$$\int_{[0,T]\times S^1} |\nabla\hat{\varphi}|^2\, dt d\theta \le 2\pi + cT^{-1}.$$

This gives the upper bound since we may extend $\hat{\varphi}$ to $[-T, T] \times S^1$ so that it is invariant under the identification hence defined on M.

To get the lower bound we use essentially the same argument as for the annulus. We consider the measure $\hat{\lambda}\, d\theta$ as above. Because of the identification we have that $\hat{\lambda}(\theta) = \hat{\lambda}(\theta+\pi)$ and the integral of $\hat{\lambda}$ is equal to 2. We take the weak limit of these measures for a sequence $T_i \to 0$, and observe that if the weak limit is not a pair of point masses we have σ_1 tending to 0, while if it is a pair of antipodal point masses then σ is bounded above by 2π plus a term which tends to 0. In either case we get a contradiction and we have finished the proof. $\square$

We now address the general case and prove the following theorem.

Theorem 4.4. *Let M be a smooth surface of genus 0 with $k \ge 2$ boundary components. If $\lambda > \sigma^*(0, k-1)$, then there is a $\delta > 0$ depending on λ such that if g is a smooth metric on M with $\sigma_1 L \ge \lambda$ then the injectivity radius of g_0 is at least δ where g_0 is the unique constant curvature metric which parametrizes the conformal class of g.*

Proof. We have done the case $k = 2$ in Proposition 4.3, so we now treat the cases $k \ge 3$. We will prove the theorem by contradiction assuming that we have a sequence of hyperbolic metrics $g_{0,i}$ on the doubled surface and a sequence of metrics g_i conformal to $g_{0,i}$ and with $L_{g_i}(\partial M) = 1$ and $\sigma_1(g_i) \ge \lambda$. We assume that the injectivity radius δ_i of $g_{0,i}$ on the doubled surface tends to 0, and we take a subsequence so that the metrics $g_{0,i}$ converge in C^2 norm on compact subsets to a complete hyperbolic metric g_0 on a surface M_0 with finite area. The surface $\tilde{M}_0$ is the limit of the surfaces $(\tilde{M}, g_{0,i})$ after a finite collection of disjoint simple closed curves have been pinched to curves of 0 length. The surface $\tilde{M}_0$ is the double of a surface M_0 which is a (possibly disconnected) hyperbolic surface with finite area and (possibly noncompact) boundary. Each curve which is pinched corresponds to two ends of M_0. There are two possibilities depending on whether a pinched curve lies in M or crosses the boundary of M. The surface M_0 is disconnected if and only if there is a pinched curve in M which is not a boundary component. The hyperbolic metric on an end corresponding to the pinching of a curve in M is given in suitable coordinates on $(t_0, \infty) \times S^1$ by $dt^2 + e^{-2t} d\theta^2$. We refer to such an end as a *cusp*. If the pinched curve γ crosses ∂M, then it must be invariant under the reflection of $\tilde{M}$ across ∂M, and it is either a component of ∂M or the

reflection fixes two points of γ, and thus γ intersects exactly two boundary components. In this case the metric on the end associated with pinching γ is the same except that it is defined on $(t_0, \infty) \times (0, \pi)$ where $\theta = 0$ and $\theta = \pi$ are portions of two separate components of ∂M_0. We refer to such an end as a *half-cusp*.

We let $M_i = (M, g_{0,i})$ and we recall that near a geodesic γ which is being pinched the metric $g_{0,i}$ in a neighborhood of γ is given by $g_{0,i} = dt^2 + (e^{-t} + \epsilon e^t)^2 d\theta^2$ in suitable coordinates (t, θ) on an interval of $\mathbb{R} \times S^1$. Here $\epsilon = \epsilon_i$ is determined by the condition that $4\pi\sqrt{\epsilon} = L_i(\gamma)$ and this representation of $g_{0,i}$ is valid for $|t - t^*| \le t^* - c$ where $t^* = 1/2 \log(1/\epsilon)$ is the value of t which represents γ and c is a fixed constant. Thus as the length of γ goes to 0 the metric converges to the cusp metric $dt^2 + e^{-2t} d\theta^2$. For a large T, we let $M_{i,T}$ be the subset of M_i with boundary equal to the set $t = T$ on each end. We then have $M_{i,T}$ converging in C^2 norm to $M_{0,T}$, the corresponding subset of M_0.

We normalize $L_{g_i}(\partial M) = 1$ and by extracting a subsequence we may assume that the arclength measures converge to a limiting measure μ on compact subsets of ∂M_0. For any connected component M_0' of M_0 we show that $\mu(\partial M_0')$ is either 0 or 1. To see this we suppose that $a = \mu(\partial M_0') \in (0, 1)$. There exists a T such that $\mu(\partial M_{i,T}') > a/2$ for i sufficiently large, where M_i' is the part of M_i converging to M_0'. For a large number R and i large enough we let u be the function which is 1 on $M_{i,T}'$, decays linearly as a function of t to 0 at $t = T + R$, and is extended to M_i to be 0 outside $M_{i,T+R}'$. We then have the Dirichlet integral $E(u) \le cR^{-2}$ since the areas are uniformly bounded. On the other hand, given $\epsilon > 0$ we have the average $\bar{u} = \int_{\partial M_i} u \, ds_i \le a + \epsilon$ for i sufficiently large (where ds_i is the arclength measure for g_i). It follows that $\int_{\partial M_i} (u - \bar{u})^2 \, ds_i \ge (1 - a - \epsilon)^2 a/2$, and so

$$\sigma_1(g_i)(1 - a - \epsilon)^2 a/2 \le \sigma_1(g_i) \int_{\partial M_i} (u - \bar{u})^2 \, ds_i \le E(u) \le cR^{-2}.$$

This is a contradiction for R sufficiently large.

We have at most one connected component of M_0 with positive boundary measure, and we now show that there is exactly one such component. If this were not the case then from above we would have $\mu(\partial M_0) = 0$ and all of the arclength would concentrate at infinity in the half-cusps. Suppose we have a half-cusp such that for any large T, we have $\limsup_i L_i(\{T \le t \le 2t^* - T\}) = a > 0$. We then must have $a = 1$ since otherwise we can show that $\sigma_1(g_i)$ becomes arbitrarily small as in the previous paragraph. Thus we are left with the situation where $\lim_i L_i(\{T \le t \le 2t^* - T\}) = 1$ for some half-cusp end. In this case we claim that $\limsup_i \sigma_1(g_i) \le 2\pi$. This follows from the argument used in Proposition 4.3; that is, we extend the arclength measure ds_i to be zero at $t = T$ and $t = 2t^* - T$ and we take a balanced conformal map φ from the simply connected region $[T, 2t^* - T] \times [0, \pi]$ to the unit disk. Since the total measure is nearly 1 and it is concentrating away from the ends we can argue that the map φ is close to constants near the ends. We may then extend it to a map $\hat{\varphi} : M_i \to D$ which is zero away from the end, which is nearly balanced, has L^2 norm near 1, and Dirichlet integral near that of φ (which is 2π). Using $\hat{\varphi}$ minus its average as a test function we then can show that $\sigma_1(g_i)$ is bounded above by a number arbitrarily close to 2π, a contradiction.

We have shown that there is a unique component M_0' of M_0 such that $\mu(\partial M_0') = 1$. We now show that M_0' has no cusps and is therefore all of M_0. Suppose M_0' has a cusp, and consider the curve γ which is being pinched to form the cusp. For i sufficiently large we modify M_i by cutting along γ. If γ is an interior curve of M, then we consider the connected component

of $M \setminus \gamma$ which contains most of the boundary measure. We now fill in the curve γ with a disk of circumference $L_i(\gamma)$ and extend the metric $g_{0,i}$ to form a surface $\hat{M}_i$ of genus 0 with l boundary components where $l < k$. Whether γ is an interior or boundary curve we have $L_i(\partial \hat{M}_i) \to 1$ as i tends to infinity. For a small $r > 0$ we consider the surface $\hat{M}_i \setminus D_r$ where D_r is a disk in a constant curvature metric concentric with the disk we added to form $\hat{M}_i$. We let $\sigma_1(i, r)$ denote the infimum of the Steklov Rayleigh quotient for $\hat{M}_i \setminus D_r$ with competitors of zero average over $\partial \hat{M}_i$ with respect to ds_i and which are 0 on ∂D_r. Since for i large, any such function can be extended to M_i to be 0 outside ∂D_r, we see that $\sigma_1(i, r) \geq \sigma_1(g_i)$ for any $r > 0$ and i sufficiently large. We will show that $\sigma_1(i, r) \leq \sigma_1(\hat{M}_i) + c|\log r|^{-1/2}$. This in turn implies $\sigma_1(g_i) \leq \sigma_1(i, r) \leq \sigma^*(0, k-1) + \epsilon$ for any given $\epsilon > 0$ if r is chosen small enough and i is chosen large enough. For ϵ small this contradicts the assumption $\sigma_1(M_i) \geq \lambda > \sigma^*(0, k-1)$.

To verify the inequality $\sigma_1(i, r) \leq \sigma_1(\hat{M}_i) + c|\log r|^{-1/2}$ we observe that $\sigma_1(i, r)$ is continuous for $r \in [0, r_0)$ ($r_0 > 0$ and fixed), monotone increasing in r, and with $\sigma_1(i, 0) = \sigma_1(\hat{M}_i)$. We let $u_{i,r}$ be a first eigenfunction of $\hat{M}_i \setminus D_r$ (with eigenvalue $\sigma_1(i, r)$) normalized to have L^2 norm 1 on $\partial \hat{M}_i$. We claim that for r small and any T, $u_{i,\rho}$ is pointwise uniformly bounded on compact subsets of $\hat{M}_{i,T} \setminus \partial \hat{M}_{i,T}$ for all $\rho \leq r$ and i large enough. In particular, $u_{i,\rho}$ is uniformly bounded near the disk we added to form $\hat{M}_i$. To obtain the bound we will need to show that $\sigma_1(i, \rho)$ is bounded from below by a positive constant for i fixed. We show that if $\sigma(i, \rho)$ is bounded below for some ρ, then we get the quantitative lower bound $\sigma_1(i, r) \leq \sigma_1(i, \rho) + c|\log r|^{-1/2}$. The result then follows by letting ρ tend to 0. To see this bound for a given ρ, we observe that on any component Ω of the complement of the zero set of $u_{i,\rho}$, the function $u_{i,\rho}$ is a *first* eigenfunction for the eigenvalue problem in Ω with Dirichlet boundary condition on $\partial\Omega \cap \hat{M}_i$ and Steklov boundary condition on $\Omega \cap \partial \hat{M}_i$. In other words the function $u_{i,\rho}$ on Ω minimizes the Rayleigh quotient

$$\frac{\int_\Omega |\nabla v|^2}{\int_{(\partial \hat{M}_i) \cap \bar{\Omega}} v^2 \, ds_i}$$

over functions v which are 0 on $\partial\Omega \cap \hat{M}_i$. Moreover the minimum value of this ratio is $\sigma_1(i, \rho)$. This follows directly from the variational characterization of $\sigma_1(\hat{M}_i)$. From here we see that no such component can be too large. First, for T large enough, the set $\hat{M}_{i,T}$ cannot be contained in such an Ω. This follows from the fact that for T large, the length $L_i(\partial \hat{M}_{i,T/2})$ is greater than $1/2$ (in fact is near 1). We can then choose a function v which is 1 on $\hat{M}_{i,T/2}$ and zero outside $\hat{M}_{i,T}$ with Dirichlet integral bounded by cT^{-2}. This contradicts the assumption that $\sigma_1(i, \rho)$ is bounded from below by a positive constant. It follows that there are points in $\partial \hat{M}_{i,T}$ at which $u_{i,\rho} = 0$. If we can find a point a fixed distance from $\partial \hat{M}_{i,T}$ at which $u_{i,\rho}$ is bounded, the result then follows from bounds on harmonic functions with bounded Dirichlet integral. To find such a point we assert that there is a point $p \in \partial \hat{M}_{i,T}$ and a fixed $\delta > 0$ with the property that each half circle of radius τ centered at p for $0 < \tau \leq \delta$ intersects the zero set of $u_{i,\rho}$. Indeed if this were not true it would follow that each component of the zero set of $u_{i,\rho}$ is contained in a small disk $D_\delta(p)$ centered at a boundary point. Since the zero set divides into two components (Courant nodal domain theorem), it follows that there can be at most one such disk, and therefore there is a component Ω of the complement of

the zero set which contains $\hat{M}_{i,T} \setminus D_\delta(p)$. Since for δ small the arclength of the interval of radius δ about p on the boundary is small, we can construct a function which is 1 away from p, vanishes in $D_\delta(p)$ and has small Rayleigh quotient, contradicting the lower bound on $\sigma_1(i,\rho)$. Therefore $u_{i,\rho}$ is locally bounded in $\hat{M}_{i,T} \setminus \partial\hat{M}_{i,T}$ as claimed.

We now choose a Lipschitz radial function ζ which is 1 outside $D_{\sqrt{r}}$ and 0 inside D_r with Dirichlet integral $E(\zeta) \le c|\log r|^{-1}$. We then let $v = \zeta u_{i,\rho}$ and estimate the Dirichlet integral

$$E(v) \le (1+\epsilon)E(u_{i,\rho}) + c(1+\epsilon^{-1})E(\zeta)$$

for any $\epsilon > 0$ where we have used the arithmetic-geometric mean inequality and the bound on $u_{i,\rho}$. We take $\epsilon = |\log r|^{-1/2}$ and conclude $E(v) \le \sigma_1(i,\rho) + c|\log r|^{-1/2}$, and this implies the bound $\sigma_1(i,r) \le \sigma_1(i,\rho) + c|\log r|^{-1/2}$ whenever $\sigma_1(i,\rho)$ has a lower bound. Since for r fixed and i large enough this assumption holds for $\rho = r$ we can then get the lower bound and the estimate for all $\rho > 0$. Letting ρ tend to 0 we obtain the desired bound $\sigma_1(i,r) \le \sigma_1(\hat{M}_i) + c|\log r|^{-1/2}$ as claimed. Thus we have shown that M_0 is connected and has no cusps.

Finally we finish the proof by showing that M_0 has no half-cusps and is therefore compact. Assume we have a half-cusp in M_0 and consider the surfaces M_i converging to M_0. Note that we have two half-cusps for each pinched curve, one on each side of the pinched curve. For i large, let (t,θ) be the coordinates described above on the half-cusp. We have shown that the measure concentrates away from the neck, so for any $\epsilon > 0$ we may choose T_0 large enough that $L_i(\{t \ge T_0\}) < \epsilon$. We form a surface M_1 with $k-1$ boundary components by making a cut along the $t = t^*,\ 0 \le \theta \le \pi$ curve. We take the boundary measure μ on ∂M_1 to be the original measure ds_i on $\{t \le T_0\}$ and 0 on the remainder of the boundary. Thus we have $\mu(\partial M_1) \ge 1-\epsilon$. We let $\sigma_1(M_1)$ be the corresponding first eigenvalue and let u_1 be a first eigenfunction normalized to have boundary L^2-norm equal to 1 with respect to μ. From above we know that u_1 is bounded by a constant depending on T_0 for points of $\{t = T_0\}$ away from the boundary points. By Fubini's theorem we may then choose $T_1 \in [T_0, T_0+1]$ so that $\int_{\{t=T_1\}} |\nabla u_1|^2 ds \le c$, and therefore we have $|u_1(T_1,\theta)| \le c$ for $0 \le \theta \le \pi$. Since u_1 satisfies the homogeneous Neumann boundary condition for $t \ge T_1$, it follows from the maximum principle that $|u_1| \le c$ on all of M_1. For $T > T_1$ to be chosen sufficiently large we choose a Lipschitz function $\zeta(t)$ which is 1 for $t \le T$, 0 for $t \ge T+1$, and with $|\zeta'| \le 1$. We then set $v = \zeta u_1$ and we extend v to M_i to be 0 for $t \ge T+1$ (on both sides of the neck being pinched). Using the bound on u_1 we may estimate the Dirchlet integral of v on M_i by $E(v) \le (1+\epsilon_1)E(u_1) + c(1+\epsilon_1^{-1})e^{-T}$ for any $\epsilon_1 > 0$. Thus we may choose T sufficiently large so that $E(v) \le E(u_1) + \epsilon$ for the ϵ chosen above. We let $\bar{v} = \int_{\partial M} v\ ds_i$, and we have

$$\sigma_1(M_i)\big(\int_{\partial M} v^2\ ds_i - \bar{v}^2\big) \le E(v) \le E(u_1) + \epsilon = \sigma_1(M_1) + \epsilon.$$

From the choice of μ and the normalization of u_1 we have by the Schwarz inequality

$$\int_{\partial M} v^2\ ds_i = 1 + \int_{\{t \ge T_0\}} v^2\ ds_i \ge 1 + (\int_{\{t \ge T_0\}} v\ ds_i)^2.$$

On the other hand we have $\bar{v} = \int_{\{t \ge T_0\}} v\ ds_i$ since $v = u_1$ integrates to 0 on $\{t \le T_0\}$. Therefore we have

$$\sigma_1(M_i) \le \sigma_1(M_1) + \epsilon \le \mu(\partial M_1)^{-1}\sigma^*(0,k-1) + \epsilon$$

since M_1 has $k-1$ boundary components. This implies

$$\sigma_1(M_i) \leq (1-\epsilon)^{-1}\sigma^*(0, k-1) + \epsilon,$$

and this is a contradiction for ϵ small. We have shown that M_0 is compact, and this is a contradiction to our assumption that the injectivity radius of $g_{0,i}$ was converging to 0. We have completed the proof of Theorem 4.4. □

5. The structure of extremal metrics

In this section we establish the connection between extremal metrics and minimal surfaces that are free boundary solutions in the ball. Let M be a compact surface with boundary, and assume g_0 is a metric on M with

$$\sigma_1(g_0)L_{g_0}(\partial M) = \max_g \sigma_1(g)L_g(\partial M)$$

where the max is over all smooth metrics on M. Let $g(t)$ be a family of smooth metrics on M with $g(0) = g_0$ and $\frac{d}{dt}g(t) = h(t)$, where $h(t) \in S^2(M)$ is a smooth family of symmetric $(0,2)$-tensor fields on M. Denote by $V_{g(t)}$ the eigenspace associated to the first nonzero Steklov eigenvalue $\sigma_1(t)$ of $(M, g(t))$. Define a quadratic form Q_h on smooth functions u on M as follows

$$Q_h(u) = -\int_M \langle \tau(u), h\rangle \, da_t - \frac{\sigma_1(t)}{2}\int_{\partial M} u^2 h(T,T)\, ds_t,$$

where T is the unit tangent to ∂M for the metric g_0, and where $\tau(u)$ is the stress-energy tensor of u with respect to the metric $g(t)$,

$$\tau(u) = du \otimes du - \frac{1}{2}|\nabla u|^2 g.$$

Lemma 5.1. *$\sigma_1(t)$ is a Lipschitz function of t, and if $\dot\sigma_1(t_0)$ exists, then*

$$\dot\sigma_1(t_0) = Q_h(u)$$

for any $u \in V_{g(t_0)}$ with $||u||_{L^2} = 1$.

Proof. To see that $\sigma_1(t)$ is Lipschitz for small t, let $t_1 \neq t_2$ and assume without loss of generality that $\sigma_1(t_1) \leq \sigma_1(t_2)$. Now let u be a first Steklov eigenfunction for $g(t_1)$ normalized so that $\int_{\partial\Sigma} u^2\, ds_{t_1} = 1$. It then follows easily from the fact that the path $g(t)$ is smooth that

$$|\int_\Sigma |\nabla^{t_1}u|^2\, da_{t_1} - \int_\Sigma |\nabla^{t_2}u|^2\, da_{t_2}| \leq c|t_1 - t_2|$$

and

$$|\int_{\partial\Sigma} u^2\, ds_{t_1} - \int_{\partial\Sigma} u^2\, ds_{t_2}| \leq c|t_1 - t_2|, \quad |\bar u(t_1) - \bar u(t_2)| \leq c|t_1 - t_2|$$

where $\bar u(t)$ denotes the average of u over $\partial\Sigma$ with respect to the metric $g(t)$. Therefore we have

$$|\sigma_1(t_1) - \sigma_1(t_2)| = \sigma_1(t_2) - \sigma_1(t_1) \leq \frac{\int_\Sigma |\nabla^{t_2}u|^2\, da_{t_2}}{\int_{\partial\Sigma}(u - \bar u(t_2))^2\, ds_{t_2}} - \int_\Sigma |\nabla^{t_1}u|^2\, da_{t_1} \leq c|t_1 - t_2|$$

and $\sigma_1(t)$ is Lipschitz.

Choose $u_0 \in V_{g(t_0)}$ and a family of functions $u(t)$ such that $u(t_0) = u_0$ and $\int_{\partial M} u(t)\, ds_t = 0$; e.g. $u(t) = u_0 - \int_{\partial M} u_0\, ds_t$. Let

$$F(t) = \int_M |\nabla u(t)|^2\, da_t - \sigma_1(t) \int_{\partial M} u^2(t)\, ds_t.$$

Then $F(t) \geq 0$, and $F(t_0) = 0$, and we have $\dot{F}(t_0) = 0$. Differentiating F with respect to t at $t = t_0$ we therefore obtain

$$\begin{aligned} &\int_M [\, 2\langle \nabla u_0, \nabla \dot{u}_0 \rangle - \langle du_0 \otimes du_0 - \frac{1}{2}|\nabla u_0|^2 g, h\rangle \,]\, da_{t_0} \\ &\qquad = \dot{\sigma_1}(t_0) \int_{\partial M} u_0^2\, ds_{t_0} + \sigma_i(t_0) \int_{\partial M} [\, 2u_0 \dot{u}_0 + \frac{1}{2} u_0^2 h(T,T) \,]\, ds_{t_0}. \end{aligned}$$

Since u_0 is a first Steklov eigenfunction, we have

$$\int_M \langle \nabla u_0, \nabla \dot{u}_0 \rangle\, da_{t_0} = \sigma_1(t_0) \int_{\partial M} u_0\, \dot{u}_0\, ds_{t_0}.$$

Using this, and if we normalize u_0 so that $||u_0||_{L^2} = 1$, we have

$$\dot{\sigma}_1(t_0) = -\int_M \langle du_0 \otimes du_0 - \frac{1}{2}|\nabla u_0|^2 g, h\rangle\, da_{t_0} - \frac{\sigma_1(t_0)}{2} \int_{\partial M} u_0^2 h(T,T)\, ds_{t_0} = Q_h(u_0)$$

□

Proposition 5.2. *If M is a surface with boundary, and g_0 is a metric on M with*

$$\sigma_1(g_0) L_{g_0}(\partial M) = \max_g \sigma_1(g) L_g(\partial M)$$

where the max is over all smooth metrics on M. Then there exist independent first eigenfunctions $u_1, \dots, u_n$ which give a conformal minimal immersion $u = (u_1, \dots, u_n)$ of M into the unit ball B^n such that u(M) is a free boundary solution, and up to rescaling of the metric u is an isometry on ∂M.

The outline of the proof of the proposition follows an argument of Nadirashvili ([N], Theorem 5) and El Soufi and Ilias ([EI2], Theorem 1.1).

Consider the Hilbert space $\mathcal{H} = L^2(S^2(M)) \times L^2(\partial M)$, the space of pairs of L^2 symmetric $(0,2)$-tensor fields on M and L^2 functions on boundary of M.

Lemma 5.3. *Assume g_0 satisfies the conditions of Proposition 5.2, and that g_0 is rescaled so that $\sigma_1(g_0) = 1$. Then, for any $(\omega, f) \in \mathcal{H}$ with $\int_{\partial M} f\, ds = 0$, there exists $u \in V_{g_0}$ with $||u||_{L^2} = 1$ such that $\langle (\omega, f), (\tau(u), u^2) \rangle_{L^2} = 0$.*

Proof. Let $(\omega, f) \in \mathcal{H}$, and assume that $\int_{\partial M} f\, ds = 0$. Since $C^\infty(S^2(M)) \times C^\infty(M)$ is dense in $L^2(S^2(M)) \times L^2(M))$, we can approximate (ω, f) arbitrarily closely in L^2 by a smooth pair $(h, \tilde{f})$ with $\int_{\partial M} \tilde{f}\, ds = 0$. We may redefine h in a neighbourhood of the boundary to a smooth tensor whose restriction to ∂M is equal to the function $\tilde{f}$, and such that the change in the L^2 norm is arbitrarily small. In this way, we obtain a smooth sequence h_i with $\int_{\partial M} h_i(T,T)\, ds = 0$, such that $(h_i, h_i(T,T)) \to (\omega, f)$ in L^2.

Let $g(t) = \frac{L_{g_0}(\partial M)}{L_{g_0+th_i}(\partial M)}(g_0 + th_i)$. Then $g(0) = g_0$, $L_{g(t)}(\partial M) = L_{g_0}(\partial M)$, and since

$$\frac{d}{dt}\Big|_{t=0} L_{g_0+th_i}(\partial M) = \int_{\partial M} h_i(T,T)\, ds = 0$$

we have $\frac{dg}{dt}\big|_{t=0} = h_i$. Given any $\varepsilon > 0$, by the fundamental theorem of calculus,

$$\int_{-\varepsilon}^{0} \dot\sigma_1(t)\, dt = \sigma_1(0) - \sigma_1(-\varepsilon) \geq 0$$

by the assumption on g_0. Therefore there exists t, $-\varepsilon < t < 0$, such that $\dot\sigma_1(t)$ exists and $\dot\sigma_1(t) \geq 0$. Let t_j be a sequence of points with $t_j < 0$ and $t_j \to 0$, such that $\dot\sigma_1(t_j) \geq 0$. Choose $u_j \in V_{g(t_j)}$ with $||u_j||_{L^2} = 1$. Then, after passing to a subsequence, u_j converges in $C^2(M)$ to an eigenfunction $u_-^{(i)} \in V_{g_0}$ with $||u_-^{(i)}||_{L^2} = 1$. Since $Q_{h_i}(u_j) = \dot\sigma_1(t_j) \geq 0$, it follows that $Q_{h_i}(u_-^{(i)}) \geq 0$. By a similar argument, taking a limit from the right, there exists $u_+^{(i)} \in V_{g_0}$ with $||u_+^{(i)}||_{L^2} = 1$, such that $Q_{h_i}(u_+^{(i)}) \leq 0$.

After passing to subsequences, $u_+^{(i)} \to u_+$ and $u_-^{(i)} \to u_-$ in $C^2(M)$, and

$$\begin{aligned} \langle(\omega, f), (\tau(u_+), u_+^2)\rangle_{L^2} &= \lim_{i\to\infty} Q_{h_i}(u_+^{(i)}) \leq 0 \\ \langle(\omega, f), (\tau(u_-), u_-^2)\rangle_{L^2} &= \lim_{i\to\infty} Q_{h_i}(u_-^{(i)}) \geq 0. \end{aligned}$$

□

Proof of Proposition 5.2. Without loss of generality, rescale the metric g_0 so that $\sigma_1(g_0) = 1$. Let K be the convex hull in $\mathcal{H}$ of

$$\{\, (\tau(u), u^2) \,:\, u \in V_{g_0}\}.$$

We claim that $(0,1) \in K$. If $(0,1) \notin K$, then since K is a convex cone which lies in a finite dimensional subspace, the Hahn-Banach theorem implies the existence of $(\omega, f) \in \mathcal{H}$ that separates $(0,1)$ from K; in particular such that

$$\begin{aligned} \langle(\omega, f), (0,1)\rangle_{L^2} &> 0, \quad \text{and} \\ \langle(\omega, f), (\tau(u), u^2)\rangle_{L^2} &< 0 \quad \text{for all } u \in V_{g_0} \setminus \{0\}. \end{aligned}$$

Let $\tilde f = f - \int_{\partial M} f\, ds$. Then, $\int_{\partial M} \tilde f\, ds = 0$, and

$$\begin{aligned} \langle(\omega, \tilde f), (\tau(u), u^2)\rangle_{L^2} &= \int_M \langle \omega, \tau(u)\rangle\, da + \int_{\partial M} \tilde f u^2\, ds \\ &= \int_M \langle \omega, \tau(u)\rangle\, da + \int_{\partial M} f u^2\, ds - \frac{\int_{\partial M} f}{L_{g_0}(\partial M)} \int_{\partial M} u^2\, ds \\ &= \langle(\omega, f), (\tau(u), u^2)\rangle_{L^2} - \frac{\int_{\partial M} u^2\, ds}{L_{g_0}(\partial M)} \langle(\omega, f), (0,1)\rangle_{L^2} \\ &< 0. \end{aligned}$$

This contradicts Lemma 5.3. Therefore, $(0, 1) \in K$, and since K is contained in a finite dimensional subspace, there exist independent eigenfunctions $u_1, \dots, u_n \in V_{g_0}$ such that

$$0 = \sum_{i=1}^{n} \tau(u_i) = \sum_{i=1}^{n} (du_i \otimes du_i - \frac{1}{2}|\nabla u_i|^2 g_0) \quad \text{on } M$$

$$1 = \sum_{i=1}^{n} u_i^2 \quad \text{on } \partial M$$

Thus $u = (u_1, \dots, u_n) : M \to B^n$ is a conformal minimal immersion. Since u_i is a first Steklov eigenfunction and $\sigma_1(g_0) = 1$, we have $\frac{\partial u_i}{\partial \eta} = u_i$ on ∂M for $i = 1, \dots, n$. Therefore,

$$\left| \frac{\partial u}{\partial \eta} \right|^2 = |u|^2 = 1 \qquad \text{on } \partial M,$$

and since u is conformal, u is an isometry on ∂M. □

6. Uniqueness of the critical catenoid

In this section we will show that the critical catenoid is the only minimal annulus in B^n which is a free boundary solution with the coordinate functions being first eigenfunctions. Recall that the critical catenoid is the unique portion of a suitably scaled catenoid which defines a free boundary surface in B^3 (see section 3 of [FS]). To this end, suppose $\Sigma = \varphi(M)$ is a free boundary solution in B^n where M is the Riemann surface $[-T, T] \times S^1$ and φ is a conformal harmonic map. We denote the coordinates on M by (t, θ), and we consider the conformal Killing vector field $X = \frac{\partial \varphi}{\partial \theta}$ defined along Σ. We will show that X is the restriction of a Killing vector field of $\mathbb{R}^n$, and hence Σ is a surface of revolution which must be the critical catenoid. We observe that since X is a conformal Killing vector field it satisfies the conditions

$$D_{e_1}X \cdot e_2 = -D_{e_2}X \cdot e_1, \quad D_{e_1}X \cdot e_1 = D_{e_2}X \cdot e_2$$

for any orthonormal basis e_1, e_2 of the tangent space.

We will have need to consider nontangential vector fields satisfying similar conditions. We call a (not necessarily tangential) vector field V a *conformal vector field* if

$$D_{e_1}V \cdot e_2 = -D_{e_2}V \cdot e_1, \quad D_{e_1}V \cdot e_1 = D_{e_2}V \cdot e_2$$

for any oriented orthonormal basis e_1, e_2 of the tangent space of Σ. A consequence is that if v is any unit vector in the tangent space the expression $D_v V \cdot v$ is constant. This may be seen by writing $v = \cos\theta e_1 + \sin\theta e_2$ and calculating

$$D_v V \cdot v = \cos^2\theta \; D_{e_1}V \cdot e_1 + \sin^2\theta \; D_{e_2}V \cdot e_2 + \sin\theta\cos\theta(D_{e_1}V \cdot e_2 + D_{e_2}V \cdot e_1)$$

and this is equal to $D_{e_1}V \cdot e_1$ for any choice of v.

The following result applies generally to free boundary solutions $\Sigma = \varphi(M)$ in B^n for any surface M. For vector fields V, W defined along Σ (but not necessarily tangent to Σ) and tangent to S^{n-1} along $\partial\Sigma$ we consider the quadratic form

$$Q(V, W) = \int_{\Sigma} \langle DV, DW \rangle \; da - \int_{\partial\Sigma} V \cdot W \; ds.$$

The following lemma relates the second variations of energy and area.

Lemma 6.1. *If φ_s is a family of maps from M to B^n with $\dot\varphi = V$, then*

$$Q(V,V) = \frac{1}{2}\frac{d^2}{ds^2}E(\varphi_s) \quad at \quad s=0.$$

If V is a conformal vector field, then $Q(V,V) = S(V^\perp, V^\perp)$.

Proof. By direct calculation we have

$$\frac{1}{2}\ddot E = \int_\Sigma (\|DV\|^2 + D\varphi\cdot D\ddot\varphi)\ da.$$

Integrating the second term by parts using the harmonicity of φ and the free boundary condition we have

$$\frac{1}{2}\ddot E = \int_\Sigma \|DV\|^2\ da + \int_{\partial\Sigma} x\cdot\ddot\varphi\ ds.$$

Since $\varphi_t(x)$ is a curve on S^{n-1} for fixed $x\in\partial\Sigma$, the normal component of the acceleration is the second fundamental form of S^{n-1} in the direction $V(x)$; thus we have

$$\frac{1}{2}\ddot E = \int_\Sigma \|DV\|^2\ da - \int_{\partial\Sigma}\|V\|^2\ ds = Q(V,V).$$

Now assume that V is a conformal vector field and consider a variation φ_s with $\dot\varphi = V$. We work in local conformal coordinates (t,θ) on M and we let

$$g_{11} = \|\varphi_t\|^2,\quad g_{22} = \|\varphi_\theta\|^2,\quad \text{and}\quad g_{12} = \varphi_t\cdot\varphi_\theta.$$

We then have $E = \int_M (g_{11}+g_{22})\ dtd\theta$ and $A = \int_M \sqrt{g_{11}g_{22}-g_{12}^2}\ dtd\theta$. The condition that V is conformal implies that at $s=0$ we have $\dot g_{11} = \dot g_{22}$ and $\dot g_{12} = 0$. We know that the second variation of area is given by $\ddot A = S(V^\perp, V^\perp)$. At $s=0$ we have $g_{11} = g_{22} = \lambda$ and $g_{12} = 0$. We compute

$$\dot A = \frac{1}{2}\int_M (g_{11}g_{22}-g_{12}^2)^{-1/2}(\dot g_{11}g_{22} + g_{11}\dot g_{22} - 2g_{12}\dot g_{12})\ dtd\theta.$$

Taking a second derivative and setting $s=0$ we obtain

$$\ddot A = \frac{1}{2}\int_M \big[-\frac{1}{2}\lambda^{-3}(\lambda\dot g_{11} + \lambda\dot g_{22})^2 + 2\lambda^{-1}\dot g_{11}\dot g_{22} + (\ddot g_{11} + \ddot g_{22})\big]\ dtd\theta.$$

The conditions on V imply that the first two terms cancel and we have

$$\ddot A = \frac{1}{2}\int_M (\ddot g_{11} + \ddot g_{22})\ dtd\theta = \frac{1}{2}\ddot E,$$

and therefore $Q(V,V) = S(V^\perp, V^\perp)$ as claimed.

$\square$

We now specialize to the annulus case, and assume that $\Sigma = \varphi(M)$ is a free boundary solution in B^n where M is the Riemann surface $[-T,T]\times S^1$ and φ is a conformal harmonic map, and denote the coordinates on M by (t,θ). We observe the following properties of the vector field $X = \frac{\partial\varphi}{\partial\theta}$.

Lemma 6.2. *The vector field X is harmonic as a vector valued function on Σ. Moreover X is in the nullspace of Q in the sense that $Q(X,Y) = 0$ for any vector field Y along Σ which is tangent to S^{n-1} along $\partial\Sigma$.*

Proof. Since φ is harmonic and $X = \frac{\partial \varphi}{\partial \theta}$ it follows that X is harmonic. Thus we may integrate by parts to write

$$Q(X,Y) = \int_{\partial\Sigma} (D_x X \cdot Y - X \cdot Y)\ ds.$$

We write $Y = Y^t + Y^\perp$ as the sum of vectors tangential and normal to Σ. Since Y is tangent to S^{n-1} it follows that both Y^t and $Y^\perp$ are also tangent to S^{n-1}. Since X is perpendicular to x, we have the second fundamental form term $D_x X \cdot Y^\perp = 0$, and thus the first term becomes $D_x X \cdot Y^t$, and since X is conformal Killing this is equal to $-D_{Y^t} X \cdot x$. This term is the second fundamental form of S^{n-1} in the directions Y^t and X, and thus is equal to $X \cdot Y^t$. Since X is tangential to Σ this is equal to $X \cdot Y$, and thus $Q(X,Y) = 0$ as claimed. □

For the next two lemmas we assume that Σ is a free boundary minimal surface in B^3 with unit normal ν. The following result, which will not be used, relates the Laplacian of a conformal vector field V to the Jacobi operator.

Lemma 6.3. *Assume that V is a conformal vector field. If we let $\psi = V \cdot \nu$, then we have $\Delta V = (\Delta\psi + |A|^2\psi)\nu$. In particular, if ψ is a Jacobi field then V is harmonic.*

Proof. We do the calculation in a local orthonormal basis e_1, e_2 which is parallel at a point; thus $D_{e_i} e_j = h_{ij}\nu$ at the point. We first compute the tangential component $\Delta V \cdot e_j$. We have

$$D_{e_1} V = (D_{e_1} V \cdot e_1)\ e_1 + (D_{e_1} V \cdot e_2)\ e_2 + (D_{e_1} V \cdot \nu)\ \nu.$$

Therefore

$$D_{e_1} D_{e_1} V \cdot e_j = D_{e_1}(D_{e_1} V \cdot e_1)\ \delta_{1j} + D_{e_1}(D_{e_1} V \cdot e_2)\ \delta_{2j} - (D_{e_1} V \cdot \nu)\ h_{1j}.$$

Similarly,

$$D_{e_2} D_{e_2} V \cdot e_j = D_{e_2}(D_{e_2} V \cdot e_1)\ \delta_{1j} + D_{e_2}(D_{e_2} V \cdot e_2)\ \delta_{2j} - (D_{e_2} V \cdot \nu)\ h_{2j}.$$

Using the conformal condition on V we have

$$D_{e_1}(D_{e_1} V \cdot e_1) + D_{e_2}(D_{e_2} V \cdot e_1) = D_{e_1}(D_{e_2} V \cdot e_2) - D_{e_2}(D_{e_1} V \cdot e_2).$$

Now we have

$$D_{e_1}(D_{e_2} V \cdot e_2) = D_{e_1} D_{e_2} V \cdot e_2 + D_{e_2} V \cdot D_{e_1} e_2 = D_{e_2} D_{e_1} V \cdot e_2 + h_{12} D_{e_2} V \cdot \nu.$$

This implies

$$D_{e_1}(D_{e_2} V \cdot e_2) = D_{e_2}(D_{e_1} V \cdot e_2) - h_{22} D_{e_1} V \cdot \nu + h_{12} D_{e_2} V \cdot \nu.$$

Thus we have

$$D_{e_1}(D_{e_1} V \cdot e_1) + D_{e_2}(D_{e_2} V \cdot e_1) = -h_{22} D_{e_1} V \cdot \nu + h_{12} D_{e_2} V \cdot \nu.$$

Thus we have

$$\Delta V \cdot e_1 = -h_{22} D_{e_1} V \cdot \nu + h_{12} D_{e_2} V \cdot \nu - (D_{e_1} V \cdot \nu) h_{11} - (D_{e_2} V \cdot \nu) h_{12} = 0.$$

Similarly we have $\Delta V \cdot e_2 = 0$, and we have shown that ΔV is a normal vector field.

We calculate $\Delta V \cdot \nu$,

$$D_{e_1} D_{e_1} V \cdot \nu = (D_{e_1} V \cdot e_1) h_{11} + (D_{e_1} V \cdot e_2) h_{12} + D_{e_1}(D_{e_1} V \cdot \nu),$$

and

$$D_{e_2} D_{e_2} V \cdot \nu = (D_{e_2} V \cdot e_1) h_{12} + (D_{e_2} V \cdot e_2) h_{22} + D_{e_2}(D_{e_2} V \cdot \nu).$$

Summing these and using the conformal condition on V and minimality we have

$$\Delta V \cdot \nu = D_{e_1}(D_{e_1}V \cdot \nu) + D_{e_2}(D_{e_2}V \cdot \nu).$$

Now $D_{e_i}V \cdot \nu = \sum_{j=1}^{2}(V \cdot e_j)h_{ij} + D_{e_i}\psi$ where $\psi = V \cdot \nu$. Therefore, using the Codazzi equations and minimality we have

$$\Delta V \cdot \nu = \sum_{i,j=1}^{2} D_{e_i}(V \cdot e_j)h_{ij} + \Delta\psi.$$

Now we have $\sum_{i,j=1}^{2}(D_{e_i}V \cdot e_j)h_{ij} = 0$ by the conformal condition and minimality, so we get

$$\Delta V \cdot \nu = \Delta\psi + |A|^2\psi$$

as claimed.

□

Let $\mathcal{C}$ denote the linear span of the functions $\{\nu_1, \nu_2, \nu_3, x \cdot \nu\}$ and we observe the following.

Lemma 6.4. *If Σ is a free boundary solution in B^3 which is not a plane disk, then $\mathcal{C}$ is a four dimensional vector space of functions on Σ.*

Proof. If there is a linear relation, then there would be a $v \in S^2$ and numbers a, b, not both zero, such that $(av + bx) \cdot \nu \equiv 0$ on Σ. Thus on the boundary of Σ we would have $av \cdot \nu \equiv 0$. This implies that either $a = 0$ or v lies in the tangent plane to Σ along each component of $\partial\Sigma$. In the latter case, the tangent plane must be constant along each component of $\partial\Sigma$. This is because the position vector x lies in the tangent plane, and x can be parallel to v at only a finite number of points. It follows that the two-vector $x \wedge v$ represents the tangent plane $T_x\partial\Sigma$ at all but a finite number of points. If T is the unit tangent, we have $D_T(x \wedge v) = T \wedge v$, and this is parallel to $x \wedge v$. It follows that the tangent plane is constant along each component of $\partial\Sigma$, and hence each boundary component lies in a 2-plane through the origin. It follows from uniqueness for the Cauchy problem that the surface is a plane disk contrary to our assumption. Therefore we must have $a = 0$ and hence $x \cdot \nu \equiv 0$ on Σ. This implies Σ is a cone and hence again a plane disk since Σ is smooth. □

We will need the following existence theorem for conformal vector fields for annular free boundary solutions.

Proposition 6.5. *Assume that Σ is an annular free boundary solution in B^3. There is subspace $\mathcal{C}_1$ of $\mathcal{C}$ of dimension at least three such that for all $\psi \in \mathcal{C}_1$ there is a tangential vector field Y^t with $x \cdot Y^t = 0$ on $\partial\Sigma$ such that the vector field $Y = Y^t + \psi\ \nu$ is conformal. Furthermore we have $Q(Y, Y) = S(\psi\nu, \psi\nu)$. For brevity of notation we denote $S(\psi\nu, \psi\nu)$ by $S(\psi, \psi)$.*

Proof. For any function ψ the equations for Y^t which dictate the condition that $Y = Y^t + \psi\ \nu$ is conformal are

$$D_{\partial_t}Y^t \cdot \partial_\theta + D_{\partial_\theta}Y^t \cdot \partial_t = 2\psi h_{12} \text{ and } D_{\partial_t}Y^t \cdot \partial_t - D_{\partial_\theta}Y^t \cdot \partial_\theta = 2\psi h_{11}$$

where $\partial_t, \partial_\theta$ denote the coordinate basis and h_{ij} the second fundamental form of Σ in this basis. If we write $Y^t = u\varphi_t + v\varphi_\theta$ we have $u = |\varphi_t|^{-2}Y^t \cdot \varphi_t$ and $v = |\varphi_\theta|^{-2}Y^t \cdot \varphi_\theta$. Setting $\lambda = |\varphi_t|^2 = |\varphi_\theta|^2$ we then have

$$u_t = \lambda^{-2}\big[\lambda(D_{\partial_t}Y^t \cdot \partial_t + Y^t \cdot \varphi_{tt}) - 2(\varphi_t \cdot \varphi_{tt})(Y^t \cdot \partial_t)\big].$$

We have $Y^t \cdot \ \varphi_{tt} = \lambda^{-1}[(Y^t\cdot\varphi_t)(\varphi_{tt}\cdot\varphi_t)+(Y^t\cdot\varphi_\theta)(\varphi_{tt}\cdot\varphi_\theta)]$ and therefore

$$u_t = \lambda^{-1}D_{\partial_t}Y^t\cdot\partial_t + \lambda^{-2}[(Y^t\cdot\varphi_\theta)(\varphi_{tt}\cdot\varphi_\theta)-(Y^t\cdot\varphi_t)(\varphi_{tt}\cdot\varphi_t)].$$

Similarly we have

$$v_\theta = \lambda^{-1}D_{\partial_\theta}Y^t\cdot\partial_\theta + \lambda^{-2}[(Y^t\cdot\varphi_t)(\varphi_{\theta\theta}\cdot\varphi_t)-(Y^t\cdot\varphi_\theta)(\varphi_{\theta\theta}\cdot\varphi_\theta)]$$

Using the fact that φ is harmonic we obtain

$$u_t - v_\theta = \lambda^{-1}(D_{\partial_t}Y^t\cdot\partial_t - D_{\partial_\theta}Y^t\cdot\partial_\theta) = 2\lambda^{-1}\psi h_{11}.$$

We can similarly check that

$$u_\theta + v_t = 2\lambda^{-1}\psi h_{12}.$$

Thus if we set $f = u+\sqrt{-1}v$ and $z = t+\sqrt{-1}\theta$, the equations become

$$\frac{\partial f}{\partial \bar z} = k$$

where $k = \lambda^{-1}\psi(h_{11}+\sqrt{-1}h_{12})$. We impose the boundary condition $\Re f = 0$ on ∂M which is the condition that Y^t be tangent to S^2 along $\partial\Sigma$. If solvable the solution is unique up to a pure imaginary constant (corresponding to the vector field which is a real multiple of $\frac{\partial\varphi}{\partial\theta}$). The adjoint boundary value problem corresponds to the operator $-\frac{\partial}{\partial z}$ with boundary condition $\Im f = 0$. This has kernel the real constants, and so by the Fredholm alternative our problem is solvable if and only if $\int_M \Re k \ dtd\theta = 0$.

We now define

$$\mathcal{C}_1 = \{\psi\in\mathcal{C} : \int_M \Re(\lambda^{-1}\psi(h_{11}+\sqrt{-1}h_{12}))\ dtd\theta = 0\}$$

which is a subspace of dimension at least three. The final statement follows from Lemma 6.1. ☐

We are now in a position to prove the main theorem of the section.

Theorem 6.6. *If Σ is a free boundary minimal surface in B^n which is homeomorphic to the annulus and such that the coordinate functions are first eigenfunctions, then $n = 3$ and Σ is congruent to the critical catenoid.*

Proof. The multiplicity bound of Theorem 2.2 implies that $n = 3$. Let $X = \frac{\partial\varphi}{\partial\theta}$ be the conformal Killing vector field associated with rotations of the annulus. We first consider the case in which $\int_{\partial\Sigma} X\ ds = 0$. Since $Q(X,X) = 0$ and $\sigma_1 = 1$ it follows that the components of X are first eigenfunctions. In this case we can complete the proof by observing that X must satisfy the Steklov boundary condition

$$\frac{\partial X}{\partial t} = X\lambda$$

where $\lambda = |\varphi_t| = |\varphi_\theta|$ is the induced conformal metric. Since we have $\frac{\partial\varphi}{\partial t} = \varphi\lambda$ we have

$$\frac{\partial^2\varphi}{\partial t\partial\theta}\cdot\frac{\partial\varphi}{\partial t} = \frac{\partial\lambda}{\partial\theta} = 0.$$

It follows that λ is constant on each component of ∂M, and therefore M with the metric induced from φ is σ-homothetic to a flat annulus, and the rotationally symmetric analysis

of [FS] implies that Σ is the critical catenoid since it is the unique free boundary conformal immersion by first eigenfunctions in the rotationally symmetric case.

Now let's assume that $\int_{\partial\Sigma} X\ ds \neq 0$. Then for any $\psi \in \mathcal{C}_1$ there is a unique conformal vector field $Y(\psi)$ with $Y(\psi)\cdot\nu = \psi$ and with $(\int_{\partial\Sigma} Y(\psi)\ ds)\cdot(\int_{\partial\Sigma} X\ ds) = 0$. Furthermore the map from ψ to $Y(\psi)$ is linear. We consider the vector space

$$\mathcal{V} = \{Y(\psi) + cX :\ \psi \in \mathcal{C}_1,\ c \in \mathbb{R}\},$$

and we observe that $\mathcal{V}$ is at least four dimensional since any nonzero vector field $Y(\psi)$ has a nontrivial normal component while X is tangential to Σ.

We define a linear transformation $T : \mathcal{V} \to \mathbb{R}^3$ by $T(V) = \int_{\partial\Sigma} V\ ds$, and observe that for dimensional reasons T has a nontrivial nullspace. Thus there is a $\psi \in \mathcal{C}_1$ and $c \in \mathbb{R}$ such that $\int_{\partial\Sigma} (Y(\psi) + cX)\ ds = 0$ with $V = Y(\psi) + cX \neq 0$. From the definition of $\mathcal{C}_1$ there is a vector $v \in S^2$ and real numbers a, b so that $\psi = (av + bx)\cdot\nu$. From Lemmas 6.2 and 6.1 we have $Q(Y(\psi) + cX, Y(\psi) + cX) = Q(Y(\psi), Y(\psi)) = S(\psi,\psi)$. Now we observe that $S(\psi,\psi) = a^2 S(v\cdot\nu, v\cdot\nu) + 2abS(x\cdot\nu, v\cdot\nu) + b^2 S(x\cdot\nu, x\cdot\nu)$. Since both $x\cdot\nu$ and $v\cdot\nu$ are Jacobi fields and $x\cdot\nu = 0$ on $\partial\Sigma$, it follows from integration by parts that $S(x\cdot\nu, v\cdot\nu) = S(x\cdot\nu, x\cdot\nu) = 0$ and so $S(\psi,\psi) = a^2 S(v\cdot\nu, v\cdot\nu)$, and by Theorem 3.1 we see that $Q(Y(\psi) + cX, Y(\psi) + cX) \leq 0$ and is strictly negative unless $a = 0$. Since $\sigma_1 = 1$, we must have $Q(Y(\psi) + cX, Y(\psi) + cX) \geq 0$. Therefore $a = 0$, and $\psi = bx\cdot\nu$. It follows that the components of $V = bY(x\cdot\nu) + cX$ are eigenfunctions.

We now compute the derivative in the x-direction along $\partial\Sigma$. We have $Y(x\cdot\nu) = Y^t + (x\cdot\nu)\ \nu$. Now we have $D_x V = V$ and we observe that $D_x X$ and $D_x Y^t$ are both tangent to Σ because the second fundamental form is diagonal along $\partial\Sigma$ and both Y^t and X are parallel to the unit tangent T to $\partial\Sigma$. Therefore if we take the derivative and the dot product with ν we have $(D_x(x\cdot\nu)\ \nu)\cdot\nu = 0$ if $b \neq 0$. This implies that $D_x(x\cdot\nu) = 0$ along $\partial\Sigma$. Since $x\cdot\nu$ is also zero along $\partial\Sigma$ and $x\cdot\nu$ is a solution of the Jacobi equation it follows from uniqueness for the Cauchy problem that $x\cdot\nu \equiv 0$ on Σ. This contradiction shows that $b = 0$, and so it follows that $V = cX$ on Σ and so the components of X are first eigenfunctions. We are now in the situation discussed in the first paragraph of this proof and It follows that Σ is the critical catenoid. □

We remark that assuming the existence of a maximizing metric, which we plan to prove in a future paper, the uniqueness result above would imply a sharp eigenvalue bound for the annulus. That is, if there exists a smooth metric g on the annulus M that maximizes $\sigma_1 L$ over all metrics on M, then by Theorem 5.2 there is a branched conformal minimal immersion $\varphi : (M, g) \to B^n$ for some $n \geq 3$ by first eigenfunctions so that φ is a σ-homothety from g to the induced metric $\varphi^*(\delta)$ where δ is the euclidean metric on B^n. By the uniqueness result Theorem 6.6 above, this immersion is congruent to the critical catenoid. Thus, assuming the existence result, for any metric on the annulus M we have

$$\sigma_1 L \leq (\sigma_1 L)_{cc}$$

with equality if and only if M is σ-homothetic to the critical catenoid; in particular,

$$\sigma^*(0,2) = (\sigma_1 L)_{cc} \approx 4\pi/1.2.$$

7. Uniqueness of the critical Möbius band

In this section we show that there is a free boundary minimal embedding of the Möbius band into B^4 by first Steklov eigenfunctions, and that it is the unique free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions.

We think of the Möbius band M as $\mathbb{R} \times S^1$ with the identification $(t, \theta) \approx (-t, \theta + \pi)$.

Proposition 7.1. *There is a minimal embedding of the Möbius band M into $\mathbb{R}^4$ given by*

$$\varphi(t, \theta) = (2 \sinh t \cos \theta, 2 \sinh t \sin \theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta)$$

For a unique choice of T_0 the restriction of φ to $[-T_0, T_0] \times S^1$ defines a proper embedding into a ball by first Steklov eigenfunctions. We may rescale the radius of the ball to 1 to get the critical Möbius band. *Explicitly T_0 is the unique positive solution of* $\coth t = 2 \tanh 2t$. *Moreover, the maximum of $\sigma_1 L$ over all rotationally symmetric metrics on the Möbius band is uniquely achieved (up to σ-homothety) by the critical Möbius band, and is equal to $(\sigma_1 L)_{cmb} = 2\pi\sqrt{3}$.*

Proof. To prove this, following the approach of [FS] Section 3 for rotationally symmetric metrics on the annulus, we do explicit analysis using separation of variables to compute the eigenvalues and eigenfunctions of the Dirichlet-to-Neumann map for rotationally symmetric metrics on the Möbius band. Consider the product $[-T, T] \times S^1$ with the identification $(t, \theta) \approx (-t, \theta + \pi)$, and with metric of the form $g = f^2(t)(dt^2 + d\theta^2)$ for a positive function f such that $f(-t) = f(t)$. The outward unit normal vector at a boundary point (T, θ) is given by $\eta = f(T)^{-1} \frac{\partial}{\partial t}$. To compute the Dirichlet-to-Neumann spectrum, as in Section 3 of [FS], we separate variables and look for harmonic functions of the form $u(t, \theta) = \alpha(t)\beta(\theta)$, but here additionally satisfying the symmetry condition $u(t, \theta) = u(-t, \theta + \pi)$. We obtain solutions for each nonnegative integer n given by linear combinations of $\sinh(nt) \sin(n\theta)$ and $\sinh(nt) \cos(n\theta)$ when n is odd, and $\cosh(nt) \sin(n\theta)$ and $\cosh(nt) \cos(n\theta)$ when n is even. For $n = 0$ the solutions are constants.

In order to be an eigenfunction for the Dirichlet-to-Neumann map we must have $u_\eta = \lambda u$ on the boundary, or $f(T)^{-1} u_t = \lambda u$ at the boundary point (T, θ). For $n = 0$ we have $u(t, \theta) = a$ and $\lambda = 0$. For $n \geq 1$ odd the eigenfunctions have $\alpha(t) = a \sinh(nt)$ and the condition is

$$n f(T)^{-1} \cosh(nT) = \lambda \sinh(nT).$$

Therefore $\lambda = n f(T)^{-1} \coth(nT)$. For $n \geq 1$ even the eigenfunctions have $\alpha(t) = a \cosh(nt)$ and the condition is

$$n f(T)^{-1} \sinh(nT) = \lambda \cosh(nT).$$

Therefore $\lambda = n f(T)^{-1} \tanh(nT)$. Note that both $n f(T)^{-1} \coth(nT)$ and $n f(T)^{-1} \coth(nT)$ are increasing functions of n. Thus if we want to find the smallest nonzero eigenvalue σ_1 of the Dirichlet-to-Neumann map we need only consider $n = 1, 2$. We must have $\sigma_1 = \min\{f(T)^{-1} \coth(T), 2f(T)^{-1} \tanh(2T)\}$. If we fix the boundary length $2\pi f(T)$, we see that $2f(T)^{-1} \tanh(2T)$ is an increasing function of T and $f(T)^{-1} \coth(T)$ is a decreasing function of T. It follows that if we fix the boundary length, then $\sigma_1 L$ is maximized for $T = T_0$ where T_0 is the positive solution of $\coth(T) = 2 \tanh(2T)$. Therefore, the maximum of $\sigma_1 L$ over all rotationally symmetric metrics on the Möbius band is $2\pi \coth T_0 = 2\pi\sqrt{3}$.

The map $\varphi : [-T_0, T_0] \times S^1 \to \mathbb{R}^4$ given by

$$\varphi(t, \theta) = (2 \sinh t \cos\theta, 2\sinh t \sin\theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta)$$

is a proper conformal map into a ball by first Steklov eigenfunctions, and gives a free boundary minimal embedding into that ball. □

We now show that the critical Möbius band is the unique free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions. First we need the following analogs of Lemma 6.4 and Proposition 6.5 from section 6.

Lemma 7.2. *Suppose Σ is a free boundary minimal surface in B^n which is not a plane disk. Let*

$$\mathcal{C} = \{E_1^\perp, \ldots, E_n^\perp, x^\perp\}$$

where $E_1, \ldots, E_n$ are the standard basis vectors in $\mathbb{R}^n$, x is the position vector, and $v^\perp$ denotes the component of v normal to Σ. Then $\mathcal{C}$ is an $(n+1)$-dimensional space of vector fields along Σ.

Proof. If there is a linear relation, then there would be a $v \in S^{n-1}$ and numbers a, b, not both zero, such that $av^\perp + bx^\perp \equiv 0$ on Σ. If $a = 0$, then $x^\perp \equiv 0$ on Σ which would imply that Σ was a cone and hence a plane disk since Σ is smooth, a contradiction. Therefore, $a \neq 0$, and so $v^\perp = -\frac{b}{a}x^\perp$. Since $x^\perp = 0$ on $\partial\Sigma$ this implies that $v^\perp = 0$ on $\partial\Sigma$. Therefore v lies in the tangent space to Σ at each point of $\partial\Sigma$. Since x also lies in the tangent plane and is independent of v at all but finitely many points, the 2-vector $x \wedge v$ represents the tangent plane when they are independent. If T is a unit tangent to $\partial\Sigma$, we have $D_T(x \wedge v) = T \wedge v$ and this is parallel to $x \wedge v$. Therefore the tangent plane $T_x\Sigma$ is constant along each component of $\partial\Sigma$, and each component lies in a 2-plane through the origin. It follows from uniqueness for the Cauchy problem that Σ is a plane disk contrary to our assumption. □

We now specialize to the case where Σ is the Möbius band. That is, suppose $\Sigma = \varphi(M)$ is a free boundary solution in B^n where $M = [-T, T] \times S^1$ with the identification $(t, \theta) \approx (-t, \theta + \pi)$, and φ is a conformal harmonic map. The following existence theorem for conformal vector fields on the Möbius band is analogous to Proposition 6.5 for the annulus, and the proof is similar.

Proposition 7.3. *Assume that $\Sigma = \varphi(M)$ is a free boundary minimal Möbius band in B^n. There is subspace $\mathcal{C}_1$ of $\mathcal{C}$ of dimension at least n such that for all $V \in \mathcal{C}_1$ there is a tangential vector field Y^t with $x \cdot Y^t = 0$ on $\partial\Sigma$ such that the vector field $Y = Y^t + V$ is conformal. Furthermore we have $Q(Y, Y) = S(V, V)$.*

Proof. We lift to the oriented double cover $\widetilde{M} = [-T, T] \times S^1$ and look for a vector field $Y^t = u\varphi_t + v\varphi_\theta$ that is invariant $Y^t(t, \theta) = Y^t(-t, \theta + \pi)$ and hence descends to the quotient M. Since the lifted map φ is invariant we have $\varphi_t(t, \theta) = -\varphi_t(-t, \theta + \pi)$, $\varphi_\theta(t, \theta) = \varphi_\theta(-t, \theta + \pi)$ and so Y^t is invariant if

$$u(-t, \theta + \pi) = -u(t, \theta) \text{ and } v(-t, \theta + \pi) = v(t, \theta). \tag{7.1}$$

The equations for Y^t which dictate the condition that $Y = Y^t + V$ is conformal are

$$D_{\partial_t} Y^t \cdot \partial_\theta + D_{\partial_\theta} Y^t \cdot \partial_t = 2h_{12} \cdot V \text{ and } D_{\partial_t} Y^t \cdot \partial_t - D_{\partial_\theta} Y^t \cdot \partial_\theta = 2h_{11} \cdot V$$

where ∂_t, ∂_θ denote the coordinate basis and h_{ij} the vector-valued second fundamental form of Σ in this basis, and as in the proof of Proposition 6.5 we have

$$u_t - v_\theta = 2\lambda^{-1}h_{11}\cdot V \text{ and } u_\theta + v_t = 2\lambda^{-1}h_{12}\cdot V.$$

Thus if we set $f = u + \sqrt{-1}v$ and $z = t + \sqrt{-1}\theta$, the equations become

$$\frac{\partial f}{\partial \bar z} = k$$

where $k = \lambda^{-1}(h_{11}\cdot V + \sqrt{-1}h_{12}\cdot V)$. We impose the boundary condition $\Re f = 0$ on $\partial\widetilde{M}$ which is the condition that Y^t be tangent to S^{n-1} along $\partial\Sigma$. Furthermore, by (7.1), we require that $f(-t,\theta+\pi) = -\bar f(t,\theta)$. If solvable the solution is unique up to a pure imaginary constant (corresponding to the vector field which is a real multiple of $\frac{\partial\varphi}{\partial\theta}$).

Consider the operator

$$L : \mathcal{F}_1 \to L^2(M,\mathbb{C})$$

defined by

$$Lf = \frac{\partial f}{\partial \bar z}$$

on the domain

$$\mathcal{F}_1 = \{f \in H^1(\widetilde{M},\mathbb{C}) : f(-t,\theta+\pi) = -\bar f(t,\theta) \text{ on } \widetilde{M},\ \Re f = 0 \text{ on } \partial\widetilde{M}\}.$$

Since $\widetilde{M}$ is compact with boundary, and L is an elliptic operator with elliptic boundary condition, L is a Fredholm operator on the domain $\mathcal{F}_1$. Then,

$$\begin{aligned}\langle Lf, g\rangle &= \Re\int_{\widetilde{M}} \frac{\partial f}{\partial \bar z}\bar g\; dtd\theta \\ &= \Re\left[-\int_{\widetilde{M}} f\overline{\frac{\partial g}{\partial z}}\; dtd\theta + \frac{1}{2}\int_{\partial\widetilde{M}} f\bar g\; d\theta\right] \\ &= -\langle f, \frac{\partial g}{\partial z}\rangle + \frac{1}{2}\Re\int_{\partial\tilde M} f\bar g\, d\theta.\end{aligned}$$

Therefore the L^2-adjoint L^* of L is defined on the domain

$$\mathcal{F}_2 = \{g \in H^1(\widetilde{M},\mathbb{C}) : g(-t,\theta+\pi) = \bar g(t,\theta) \text{ on } \widetilde{M},\ \Im g = 0 \text{ on } \partial\widetilde{M}\}$$

and is given by

$$L^* g = -\frac{\partial g}{\partial z}.$$

This has kernel the real constants, and so by the Fredholm alternative our problem is solvable if and only if $\int_{\widetilde{M}} \Re k\; dtd\theta = 0$.

We now define

$$\mathcal{C}_1 = \{V \in \mathcal{C} : \int_{\widetilde{M}} \Re(\lambda^{-1}(h_{11}\cdot V + \sqrt{-1}h_{12}\cdot V))\; dtd\theta = 0\}$$

which is a subspace of dimension at least n. The final statement follows from Lemma 6.1. $\square$

We now prove the uniqueness result for the critical Möbius band. The proof is almost identical to the annulus case Theorem 6.6, however we include the details for completeness.

Theorem 7.4. *Assume that Σ is a free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions. Then $n = 4$ and Σ is the critical Möbius band.*

Proof. Let $X = \frac{\partial \varphi}{\partial \theta}$ be the conformal Killing vector field associated with rotations of the Möbius band. We first consider the case that $\int_{\partial\Sigma} X\ ds = 0$. Since $Q(X, X) = 0$ and $\sigma_1 = 1$ it follows that the components of X are first eigenfunctions. Therefore on ∂M we have

$$\frac{\partial X}{\partial t} = \frac{\partial^2 \varphi}{\partial t \partial \theta} = X\lambda$$

where $\lambda = |\varphi_t| = |\varphi_\theta|$ is the induced conformal metric. Taking the dot product with $\varphi_t = \varphi\lambda$ we obtain

$$\frac{\partial^2 \varphi}{\partial t \theta} \cdot \frac{\partial \varphi}{\partial t} = \frac{\partial \lambda}{\partial t} = 0.$$

It follows that M with the induced metric is σ-homothetic to a rotationally symmetric flat Möbius band, and therefore by Proposition 7.1, Σ must be the critical Möbius band.

Now let's assume that $\int_{\partial\Sigma} X\ ds \neq 0$. Then for any $V \in \mathcal{C}_1$ there is a unique conformal vector field $Y(V)$ with $Y(V)^\perp = V$ and with $(\int_{\partial\Sigma} Y(V)\ ds) \cdot (\int_{\partial\Sigma} X\ ds) = 0$. Furthermore the map from V to $Y(V)$ is linear. We consider the vector space

$$\mathcal{V} = \{Y(V) + cX :\ V \in \mathcal{C}_1,\ c \in \mathbb{R}\},$$

and we observe that $\mathcal{V}$ is at least $(n+1)$-dimensional since any nonzero vector field $Y(V)$ has a nontrivial normal component while X is tangential to Σ.

We define a linear transformation $T : \mathcal{V} \to \mathbb{R}^n$ by $T(W) = \int_{\partial\Sigma} W\ ds$, and observe that for dimensional reasons T has a nontrivial nullspace. Thus there is a $V \in \mathcal{C}_1$ and $c \in \mathbb{R}$ such that $\int_{\partial\Sigma} (Y(V) + cX)\ ds = 0$ with $Y(V) + cX \neq 0$. From the definition of $\mathcal{C}_1$ there is a vector $v \in S^{n-1}$ and real numbers a, b so that $V = av^\perp + bx^\perp$. From Lemmas 6.2 and 6.1 we have $Q(Y(V) + cX, Y(V) + cX) = Q(Y(V), Y(V)) = S(V, V)$. Now we observe that $S(V, V) = a^2 S(v^\perp, v^\perp) + 2ab S(x^\perp, v^\perp) + b^2 S(x^\perp, x^\perp)$. Since both $x^\perp$ and $v^\perp$ are Jacobi fields and $x^\perp = 0$ on $\partial\Sigma$, it follows from integration by parts that $S(x^\perp, v^\perp) = S(x^\perp, x^\perp) = 0$ and so $S(V, V) = a^2 S(v^\perp, v^\perp)$, and by Theorem 3.1 we see that $Q(Y(V)+cX, Y(V)+cX) \leq 0$ and is strictly negative unless $a = 0$. Since $\sigma_1 = 1$, we must have $Q(Y(V)+cX, Y(V)+cX) \geq 0$. Therefore $Q(Y(V) + cX, Y(V) + cX) = 0$, which implies that $a = 0$ so $V = bx^\perp$, and the components of $W = bY(x^\perp) + cX$ are first eigenfunctions.

We now observe that the normal component of the derivative $D_x W$ along $\partial\Sigma$ is 0 since $D_x W = W$ along $\partial\Sigma$ and $W^\perp = 0$ along $\partial\Sigma$. We have $Y(x^\perp) = Y^t + x^\perp$, and so

$$0 = (D_x W)^\perp = b(D_x Y^t)^\perp + b(D_x x^\perp)^\perp + c(D_x X)^\perp.$$

We observe that $D_x X$ and $D_x Y^t$ are both tangent to Σ because the second fundamental form is diagonal in the basis $\{x, T\}$ along $\partial\Sigma$ and both Y^t and X are parallel to the unit tangent T to $\partial\Sigma$. Therefore $b(D_x x^\perp)^\perp = 0$, and if $b \neq 0$ we have $(D_x x^\perp)^\perp = 0$. But $(D_x x^\perp)^t = 0$ on $\partial\Sigma$ since this is a second fundmental form term and $x^\perp = 0$ on $\partial\Sigma$. Therefore $D_x x^\perp = 0$ along $\partial\Sigma$. Since $x^\perp$ is also zero along $\partial\Sigma$ and $x^\perp$ is a solution of the Jacobi equation it follows from uniqueness for the Cauchy problem for the Jacobi operator that $x^\perp \equiv 0$ on Σ. This contradiction shows that $b = 0$, and so it follows that $X = c^{-1}W$ is a first eigenfunction. It now follows that Σ must be the critical Möbius band by the argument given in the first paragraph of this proof. $\square$

We remark that assuming the existence of a maximizing metric, which we plan to prove in a future paper, the uniqueness result above would imply a sharp eigenvalue bound for the Möbius band. That is, if there exists a smooth metric g on the Möbius band M that maximizes $\sigma_1 L$ over all metrics on M, then by Theorem 5.2 there is a branched conformal minimal immersion $\varphi : (M, g) \to B^n$ for some $n \geq 3$ by first eigenfunctions so that φ is a σ-homothety from g to the induced metric $\varphi^*(\delta)$ where δ is the euclidean metric on B^n. By the uniqueness result Theorem 6.6 above, this immersion is congruent to the critical Möbius band. Thus, assuming the existence result, for any metric on the Möbius band M we have

$$\sigma_1 L \leq (\sigma_1 L)_{cmb}$$

with equality if and only if M is σ-homothetic to the critical Möbius band; in particular,

$$\sigma^{\#}(0,1) = (\sigma_1 L)_{cmb} = 2\pi\sqrt{3}.$$

8. The asymptotic behavior as k goes to infinity

In this section we discuss the limit of extremal surfaces. Thus we will be considering free boundary minimal surfaces Σ_k in B^3 which are of genus zero with k boundary components. We first derive an important result concerning the geometry of such surfaces.

Proposition 8.1. *Assume that Σ_k is a branched minimal immersion in B^3 which satisfies the free boundary condition and has genus zero with $k > 1$ boundary components. Assume also that the coordinate functions are first Steklov eigenfunctions. It then follows that Σ_k is embedded, does not contain the origin 0, and is star-shaped in the sense that a ray from 0 hits Σ_k at most once. Furthermore Σ_k is a stable minimal surface with area bounded by 4π.*

Proof. From the nodal domain theorem it follows that the gradient of any first eigenfunction u is nonzero on the zero set of u. By assumption, for any unit vector v in $\mathbb{R}^3$, the function $u = x \cdot v$ is a first eigenfunction. This implies that any plane through the origin intersects Σ_k transversally, or to put it another way, the (affine) tangent plane at each point of Σ_k does not contain 0. It follows that Σ_k cannot contain the origin since otherwise its tangent plane would violate the condition.

Let ν be a choice of unit normal vector chosen so that $x \cdot \nu$ is positive at some point of Σ_k. We now claim that $x \cdot \nu$ is positive everywhere. To see this observe that if $x \cdot \nu = 0$ at a point $x \in \Sigma_k$ then the tangent plane contains the line between the origin and x. It follows that the tangent plane contains the origin and this violates the transversality condition.

It also follows that Σ_k is free of branch points since, if $x \in \Sigma_k$ were a branch point, then any linear function vanishing at x would have a critical point there, a contradiction.

We note that the zero set of a first Steklov eigenfunction u (of a genus 0 surface M) which is transverse to a boundary component Γ must intersect it in either no points or two points. To see this observe that if the zero set of u separated Γ into at least four arcs on which u alternates in sign, then the arcs on which u is positive must lie in a single connected component of the positive set of u since there are two nodal domains. Thus we can join two points p and q in separate arcs on which $u > 0$ by a path in M on which $u > 0$. This path together with one of the arcs of Γ between p and q then separates M into two connected components (since M has genus 0). But there is a negative arc of Γ in each of the components and this contradicts the fact that the negative set of u is connected.

We now show that each boundary component of Σ_k is an embedded curve. Assume we have parametrized Σ_k by an immersion φ from a domain surface M into the ball. We show that φ is an embedding on each boundary component of M. Suppose to the contrary that we had a boundary component Γ and distinct points p and q on Γ such $\varphi(p) = \varphi(q) = x_0$, then we can choose a linear function $l(x)$ vanishing at x_0 and at another chosen point x_1 of $\varphi(\Gamma)$. The function $l \circ \varphi$ is then a first Steklov eigenfunction vanishing at more than two points of Γ, a contradiction.

We now show that Σ_k is star-shaped and therefore embedded. To see this we let φ be a parametrizing immersion from a surface M. Since M has genus 0, we may take M to be a subset of S^2 whose complement consists of k disks. For each boundary curve Γ of M we have shown that φ is an embedding of Γ into S^2. Since we have chosen a unit normal ν for Σ_k by the requirement that $x \cdot \nu > 0$, we may choose the disk D_Γ bounded by $\varphi(\Gamma)$ such that ν is the outward pointing unit normal to D_Γ. We now extend φ to be an immersion of S^2 by filling each disk with a diffeomorphism to D_Γ. We may then smooth out the 90^0 corner along Γ to obtain an immersion of S^2 into B^3 with $x \cdot \nu > 0$ everywhere. It follows that the map f from S^2 to S^2 given by $f(p) = \varphi(p)/|\varphi(p)|$ is a local diffeomorphism at each point. Therefore f is a global diffeomorphism and Σ_k is star-shaped and hence embedded.

Finally, the coarse upper bound of Theorem 2.1 implies that $L(\partial\Sigma_k) \leq 8\pi$ since $\sigma_1 = 1$, and this shows that the area of Σ_k is at most 4π. □

We are now ready to prove the main convergence theorem.

Theorem 8.2. *After a suitable rotation of each Σ_k, the sequence Σ_k converges in C^3 norm on compact subsets of B^3 to the disk $\{z = 0\}$ taken with multiplicity two. Furthermore we have $\lim_{k\to\infty} A(\Sigma_k) = 2\pi$ and $\lim_{k\to\infty} L(\partial\Sigma_k) = 4\pi$.*

Proof. Since $x \cdot \nu$ is a positive Jacobi field, we know (see [FCS]) that Σ_k is a stable minimal surface. By the curvature estimates [Sc] we have a uniform bound on the second fundamental form of Σ_k on each compact subset of B^3; in fact, at all points a fixed geodesic distance from $\partial\Sigma_k$. Since the area is also bounded we have a sequence k' such that $\Sigma_{k'}$ converges in C^3 norm to a smooth minimal surface Σ_∞ possibly with multiplicity.

We show that Σ_∞ is a disk containing the origin and the multiplicity is two. Suppose the (integer) multiplicity is $m \geq 1$. We first show that the area of $\Sigma_{k'}$ converges to $mA(\Sigma_\infty)$. To see this we observe that the statement follows from the C^3 convergence on compact subsets together with uniform bounds on the area near the boundary. Specifically we can show that $A(\Sigma_k \cap (B_1 \setminus B_{1-\delta})) \leq c\delta$ for a fixed constant c. This follows from approximate monotonicity in balls around boundary points together with the global upper bound on the area. Precisely we have for $v \in \partial\Sigma_k$ and $r < 1/4$ the bound $A(\Sigma_k \cap B_r(v)) \leq cr^2$. The bound on the annular region then follows by covering the boundary with $N \approx c/\delta$ balls of radius δ (possible since $L(\partial\Sigma_k)$ is uniformly bounded). The union of the corresponding radius 2δ balls then covers $\Sigma_k \cap (B_1 - B_{1-\delta})$ and yields the area bound. We can prove the monotonicity of the weighted area ratio $e^{cr}r^{-2}A(B_r(v))$ for $0 < r \leq 1/4$ by using an appropriate vector field which is tangent to the sphere along the boundary. For example we can take the conformal vector field

$$Y = (x \cdot v)x - \frac{1 + |x|^2}{2}v.$$

For any unit vector e we have $\nabla_e Y \cdot e = x \cdot v$, and an easy estimate implies $|Y - (x - v)| \leq c|x - v|^2$. We apply the first variation formula on $\Sigma_k \cap B_r(v)$ to obtain

$$2 \int_{\Sigma_k \cap B_r(v)} x \cdot v \; da = \int_{\Sigma \cap \partial B_r(v)} Y \cdot \eta \; ds$$

where η is the unit conormal vector and there is no contribution along ∂B^3 since η is orthogonal to Y at those points. The information we have about Y then implies the differential inequality

$$(2 - cr) A(\Sigma_k \cap B_r(v)) \leq r \frac{d}{dr} A(\Sigma_k \cap B_r(v)).$$

This implies the desired monotonicity statement.

To complete the proof that $A(\Sigma_{k'}) \to mA(\Sigma_\infty)$ we need to show that for k large, all points of Σ_k which are near the boundary of B^3 are also near $\partial\Sigma_k$ and thus have small area. In particular we must show that for $\epsilon > 0$ there is a $\delta > 0$ and N such that for $k \geq N$

$$\{x \in \Sigma_k : \; |x| \geq 1 - \delta\} \subseteq \{x \in \Sigma_k : \; d(x, \partial\Sigma_k) < \epsilon\}.$$

This follows from curvature estimates by an indirect argument. Indeed, suppose there is $\epsilon > 0$ such that the conclusion fails for a sequence $\delta_i \to 0$ and $k_i \to \infty$. Then we would have a sequence of points $x_i \in \Sigma_{k_i}$ which are at least a distance ϵ from $\partial\Sigma_{k_i}$ and which converge to ∂B^3. The curvature estimates imply that a fixed neighborhood of x_i in Σ_{k_i} has bounded curvature and area, and therefore has a subsequence which converges in C^2 norm to a minimal surface in B^3. This violates the maximum principle since these surfaces have an interior point (the limit of x_i) which lies on ∂B^3. Combining this result with the boundary monotonicity we then conclude that $\lim_{i'} A(\Sigma_{i'}) = A(\Sigma_\infty)$.

Now we show that the multiplicity cannot be 1. Indeed, if $m = 1$, then Σ_∞ is a smooth embedded free boundary solution inside the ball. At each boundary point x of Σ_∞ there is a density which is an integer multiple of $1/2$ and the tangent cone at any boundary point is a half-plane with integer multiplicity. This follows from the curvature estimate which holds for a blow-up sequence at a fixed distance from the boundary and implies that the rescaled surface is locally, away from the boundary, a union of graphs over the tangent plane. By Allard-type minimal surface regularity theorems (see [GJ]), a boundary point x is a smooth point if and only if the density at x is equal to $1/2$. We use the argument of [FL] to first show that Σ_∞ is connected. Indeed if we have two connected components Σ' and Σ'', then we can find nearest points $x' \in \bar{\Sigma}'$ and $x'' \in \bar{\Sigma}''$. By the maximum principle both points lie on ∂B^3, and it follows that the density of Σ' at x' is $1/2$ as is the density of Σ'' at x''. By the boundary maximum principle (see [FL]) we get a contradiction. Therefore we conclude that Σ_∞ is connected. We do not know how to rule out the existence of a free boundary surface with infinite topology in general, but in this case since the metrics on $\Sigma_{k'}$ are converging to that of Σ_∞, we may use the argument of Proposition 4.1 to punch a hole in Σ_∞ and strictly increase $\sigma_1 L$. Because of the convergence, when we punch a corresponding hole in $\Sigma_{k'}$ for k' large we get a sequence of genus 0 surfaces with $\sigma_1 L$ converging to a number greater than $\lim_k \sigma^*(0, k)$ a contradiction. Therefore we have $m \geq 2$.

Since $m \geq 2$, it must be true that Σ_∞ contains the origin since otherwise the distance from the origin to Σ_k would be bounded from below, and the ray from the nearest point would intersect Σ_k at m points contradicting the star-shaped property. It also follows that $m = 2$ since otherwise one of the rays from the origin orthogonal to the plane $T_0\Sigma_\infty$ would

intersect Σ_k in at least two points. Now it must be true that $x \cdot \nu$ is identically zero on Σ_∞ and hence Σ_∞ is a cone (and therefore a plane since it is smooth). This follows because if $x \cdot \nu > 0$ at some point, then the ray from the origin to x intersects Σ_∞ transversally at x. This implies that the ray intersects $\Sigma_{k'}$ in two points for k' large, a contradiction. Therefore $m = 2$ and Σ_∞ is a plane.

Finally, since the subsequential limit is unique up to rotation, we may rotate each Σ_k so that the sequence Σ_k converges to $\{z = 0\}$ with multiplicity two. $\square$

Combining the previous theorem with the results of the previous section we have proven the main theorem of the section.

Theorem 8.3. *The sequence $\sigma^*(0, k)$ is strictly increasing in k and converges to 4π as k tends to infinity. For each k, if a maximizing metric exists, then it is achieved by a free boundary minimal surface Σ_k in B^3 of area less than 2π. The limit of these minimal surfaces as k tends to infinity is a double disk.*

By a blow-up analysis it is possible to say a bit more about the surfaces Σ_k near the boundary of B^3.

Remark 8.4. For large k, Σ_k is approximately a pair of nearby parallel plane disks joined by k boundary bridges each of which is approximately a scaled down version of half of the catenoid gotten by dividing with a plane containing the axis.

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